

# THE $j$ -INVARIANT OF A PLANE TROPICAL CUBIC

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ABSTRACT. Several results in tropical geometry have related the  $j$ -invariant of an algebraic plane curve of genus one to the cycle length of a tropical curve of genus one. In this paper, we prove that for a plane cubic over the field of Puiseux series the negative of the generic valuation of the  $j$ -invariant is equal to the cycle length of the tropicalization of the curve, if there is a cycle at all.

## 1. INTRODUCTION

Tropical geometry is a new and active field of research. Roughly, its main idea is to replace algebraic varieties by piece-wise linear objects called tropical varieties. These objects may be easier to deal with and new methods from combinatorics can be used to handle them. A lot of work has been done to “translate” terms and definitions to the tropical world. Sometimes a translation is justified by the appropriate use of the new tropical object rather than by an argument why this is the correct tropicalization. This is the case for the  $j$ -invariant of an elliptic curve. Many results predict that the “tropical  $j$ -invariant” of a tropical curve of genus one is its cycle length.

The  $j$ -invariant is an invariant which coincides for two smooth elliptic curves over an algebraically closed fields if and only if they are isomorphic. In [10], isomorphisms (“equivalences”) between abstract tropical curves are defined, and two elliptic abstract tropical curves are equivalent if and only if they have the same cycle length. Thus the cycle length plays the same role in the tropical setting as the  $j$ -invariant does in the algebraic setting. Also, the possibility to define a group law on the cycle of a tropical curve (see [18]) using distances indicates the importance of the cycle and its length. Furthermore, the numbers of tropical curves with fixed cycle length are in correspondence to the numbers of curves with fixed  $j$ -invariant (see [9]). But there is also a result which suggests that the cycle length actually might be the correct tropicalization of the  $j$ -invariant (as introduced on p. 4). This is a byproduct of the proof of [15, Theorem 5.4.1]. It says that given a tropical curve  $\mathcal{C}$  with cycle length  $l$  and a Puiseux series  $j$  of valuation  $-l$ , we can embed the elliptic curve with  $j$ -invariant  $j$  such that its tropicalization is equal to  $\mathcal{C}$ .

All these results indicate that the “tropical  $j$ -invariant” should be defined as the cycle length (respectively, its negative). The aim of this paper is to show that for a plane cubic the  $j$ -invariant really tropicalizes to the negative of the cycle length.

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More precisely, we define plane cubic curves over the field of Puiseux series  $\mathbb{K} = \mathbb{C}\{\{t\}\}$  and use the valuation map to tropicalize them (see p. 4). The  $j$ -invariant of an elliptic curve over the Puiseux series is a Puiseux series itself. Our main theorem (see Theorem 4.1) is that the negative of the cycle length of the tropicalization of a smooth cubic curve in  $\mathbb{P}_{\mathbb{K}}^2$  (assuming it has a cycle – see Definition 3.11) is always equal to the generic valuation of the  $j$ -invariant (see Definition 2.2) and it is actually equal to the valuation of the  $j$ -invariant itself if no terms in the  $j$ -invariant cancel – which generically is the case. A corollary (see Corollary 4.2) of this theorem is that if an elliptic curve has a  $j$ -invariant with a positive valuation, then its tropicalization does not have a cycle.

There is an intriguing similarity to bad reduction of elliptic curves over discrete valuation rings. Firstly, only elliptic curves whose  $j$ -invariant has a negative valuation have bad reduction, and secondly the negative of this valuation is then the “cycle length” of the special fiber in the Néron model of the curve in the sense that it is the number of projective lines forming the cycle. We are working on better understanding of the connection of these two results. Moreover, in a forthcoming paper we will show that the main result of this paper can be generalized to smooth elliptic curves on toric surfaces other than the projective plane.

This paper is organized as follows. In Section 2 we recall the definition of the  $j$ -invariant of a plane cubic as a rational function in the cubic’s coefficients. Its denominator is the discriminant of the cubic. Moreover, we observe that the generic valuation (see Definition 2.2) of the  $j$ -invariant is a piece-wise linear function. In Section 3 we recall basic definitions of tropical geometry and show that the function “cycle length” is piece-wise linear as well. The main theorem is stated in Section 4. As we know already that the two functions we want to compare are piece-wise linear the proof consists of two main steps: first we compare certain domains of linearity, then we compare the two linear functions on each domain. We present two proofs since we believe that each of them is interesting on its own. For the first proof, we choose as domains of linearity the union of  $\Delta$ -equivalent cones of the secondary fan of  $\mathcal{A}_3 = \{(i, j) \mid 0 \leq i, j, i + j \leq 3\}$  (i.e. cones of the Gröbner fan of the discriminant). The comparison of the two linear functions “generic valuation of the  $j$ -invariant” and “cycle length” on each such cone is done in Section 5. In the second proof, we choose smaller domains of linearity — cones of the secondary fan of  $\mathcal{A}_3$ . To compare the two linear functions, we have to classify the rays of the secondary fan of  $\mathcal{A}_3$  and compare them on each ray. Section 8 is concerned with this classification of the rays of the secondary fan of  $\mathcal{A}_3$ . In Section 6 we give an alternative proof of the fact that for an arbitrary convex lattice polytope the Gröbner fan of the discriminant is a coarsening of the secondary fan, and in Section 7 we study the numerator of the  $j$ -invariant. These two sections are important to understand the domains of linearity of the function “generic valuation of the  $j$ -invariant”.

Parts of our proofs and many examples rely on computations performed using `polymake` [3], `TOPCOM` [12] and `SINGULAR` [5]. The `SINGULAR` code that we used for this is contained in the library `jInvariant.lib` (see [8]) and it is available via the URL

<http://www.mathematik.uni-kl.de/~keilen/en/jinvariant.html>.

More detailed explanations on how to use the code can be found there.

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## 2. THE $J$ -INVARIANT AND ITS VALUATION

Since every smooth elliptic curve can be embedded into the projective plane as a cubic it makes sense to start the investigation of smooth elliptic curves and their tropicalizations by studying cubics in the plane. And since the tropicalization of a curve highly depends on its embedding we want to consider all such planar embeddings of a given curve at the same time. That is, we start with a non-zero homogeneous polynomial

$$f = \sum_{i+j=0}^3 a_{ij} x^i y^j z^{3-i-j}$$

of degree 3 as input data. Here the coefficients  $a_{ij}$  are thought of as elements of the field

$$\mathbb{K} = \bigcup_{N=1}^{\infty} \text{Quot} \left( \mathbb{C}[[t^{\frac{1}{N}}]] \right) = \left\{ \sum_{\nu=m}^{\infty} c_{\nu} \cdot t^{\frac{\nu}{N}} \mid c_{\nu} \in \mathbb{C}, N \in \mathbb{Z}_{>0}, m \in \mathbb{Z} \right\}$$

of Puiseux series over  $\mathbb{C}$ , and we consider the algebraic curve  $C = V(f) \subset \mathbb{P}_{\mathbb{K}}^2$  defined by  $f$ . Since the tropicalization of  $C$  only depends on the points of  $C$  in the torus  $(\mathbb{K}^*)^2 = \{(x : y : 1) \mid x \neq 0 \neq y\}$  (see Section 3) we may as well replace  $f$  by its affine equation

$$f = \sum_{i+j=0}^3 a_{ij} x^i y^j,$$

and we will do so for the remaining part of the paper – always keeping in mind that via homogenization  $f$  defines a cubic in the projective plane.

In our investigation the Newton polytope of  $f$ , the triangle  $Q_3$  with endpoints  $(0, 0)$ ,  $(0, 3)$  and  $(3, 0)$ , plays an important role. We denote by  $\mathcal{A}_3 := Q_3 \cap \mathbb{Z}^2$  its integer points. That way we can write the equation for  $f$  as  $f = \sum_{(i,j) \in \mathcal{A}_3} a_{ij} x^i y^j$ .

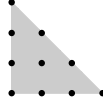


FIGURE 1.  $Q_3$  and  $\mathcal{A}_3$

If the curve  $C$  is smooth it is fixed up to isomorphism by a single invariant  $j(C) = j(f) \in \mathbb{K}$ , the  $j$ -invariant, which can be computed as a rational function, say

$$j(f) = \frac{A}{\Delta}$$

in the coefficients  $a_{ij}$  of  $f$  – now thought of as indeterminates – where  $A, \Delta \in \mathbb{Q}[a_{i,j} \mid (i,j) \in \mathcal{A}_3]$  are homogeneous polynomials of degree 12. Since we quite frequently need to refer to polynomials in the  $a_{ij}$  we introduce the convention

$\underline{a} = (a_{ij} \mid (i, j) \in \mathcal{A}_3)$  and if  $\omega \in \mathbb{N}^{\mathcal{A}_3}$  is a multi index then  $\underline{a}^\omega = \prod_{(i,j) \in \mathcal{A}_3} a_{ij}^{\omega_{ij}}$ . The denominator  $\Delta$  is actually the *discriminant* of  $f$  (see [4]).

The field  $\mathbb{K}$  of Puiseux series comes in a natural way with a *valuation*, namely

$$\text{val} : \mathbb{K}^* \rightarrow \mathbb{Q} : \sum_{\nu=m}^{\infty} c_\nu \cdot t^{\frac{\nu}{N}} \mapsto \min \left\{ \frac{\nu}{N} \mid c_\nu \neq 0 \right\},$$

and we may extend the valuation to  $\mathbb{K}$  by  $\text{val}(0) = \infty$ . If  $k = \sum_{\nu=m}^{\infty} c_\nu \cdot t^{\frac{\nu}{N}} \in \mathbb{K}$  with  $\text{val}(k) = \frac{m}{N}$  then we call  $\text{lc}(k) := c_m$  the *leading coefficient* of the formal power series  $k$ . We sometimes call  $\text{val}(k)$  the *tropicalization* of  $k$ .

Throughout the paper we will treat polynomials in the variables  $(x, y)$  as well as in the variables  $\underline{x} = (a_{ij} \mid (i, j) \in \mathcal{A}_3)$ , and many results will be derived for both cases at the same time. We therefore want to introduce a unifying notation which we use whenever we work with polynomials with an arbitrary set of variables.

### Notation 2.1

We set  $\underline{x} = (x_\lambda \mid \lambda \in \Lambda)$  where  $\Lambda$  is some finite index set. If  $\omega = (\omega_\lambda \mid \lambda \in \Lambda) \in \mathbb{N}^\Lambda$  then we use the usual multi index

$$\underline{x}^\omega = \prod_{\lambda \in \Lambda} x_\lambda^{\omega_\lambda},$$

and if  $\mathcal{A} \subset \mathbb{N}^\Lambda$  is finite and  $h_\omega \in \mathbb{K}$  for  $\omega \in \mathcal{A}$  then

$$h = \sum_{\omega \in \mathcal{A}} h_\omega \cdot \underline{x}^\omega \in \mathbb{K}[\underline{x}]$$

is a polynomial over  $\mathbb{K}$ . Later on we will sometimes need Laurent polynomials instead of polynomials and we then allow negative exponents.

It is our aim to study the valuation of the  $j$ -invariant of the curve  $C$  given by fixing values for the  $a_{ij}$ . It would be nice if it only depended on their valuations, but this is only true if the leading coefficients of the  $a_{ij}$  are sufficiently general. We therefore introduce the notion of *generic valuation* for a polynomial like  $A$  or  $\Delta$ . In order to do so we need  $t$ -initial forms, a concept which is needed in a broader context further down.

Given a polynomial  $0 \neq h = \sum_{\omega \in \mathcal{A}} h_\omega \cdot \underline{x}^\omega \in \mathbb{K}[\underline{x}]$  and a point  $v \in \mathbb{R}^\Lambda$ , we define the  $v$ -weight

$$\text{weight}_v(h) = \min\{\text{val}(h_\omega) + v \cdot \omega \mid h_\omega \neq 0\}$$

of  $h$ , where  $v \cdot \omega = \sum_{\lambda \in \Lambda} v_\lambda \cdot \omega_\lambda$ , and the  $t$ -initial form

$$\text{t-in}_v(h) = \sum_{\text{val}(h_\omega) + v \cdot \omega = \text{weight}_v(h)} \text{lc}(h_\omega) \cdot \underline{x}^\omega$$

of  $h$  with respect to  $v$ .

### Definition 2.2

If  $0 \neq h = \sum_{\omega \in \mathcal{A}} h_\omega \cdot \underline{x}^\omega \in \mathbb{Q}[\underline{x}] \subset \mathbb{K}[\underline{x}]$  has *constant rational coefficients* and  $v \in \mathbb{R}^\Lambda$ , then  $\text{val}_v(h) := \text{weight}_v(h) = \min\{v \cdot \omega \mid h_\omega \neq 0\}$  is called the *generic valuation* of  $h$  at  $v$ .

**Remark 2.3**

If  $0 \neq h \in \mathbb{Q}[\underline{x}]$ , then all  $v \in \mathbb{R}^\Lambda$  such that the  $t$ -initial forms  $\text{t-in}_v(h)$  coincide with each other form a cone and the collection of these cones forms the *Gröbner fan* of  $h$  (see e.g. [17, Chap. 2]). Top-dimensional cones correspond to  $t$ -initial forms which are monomials.

**Lemma 2.4**

If  $0 \neq h \in \mathbb{Q}[\underline{x}]$ , then the function

$$\text{val.}(h) : \mathbb{R}^\Lambda \rightarrow \mathbb{R} : v \mapsto \text{val}_v(h)$$

is piece-wise linear, and it is linear on a top-dimensional cone of the Gröbner fan of  $h$ . Moreover, if  $v \in \mathbb{R}^\Lambda$  is in the interior of a top-dimensional cone of the Gröbner fan of  $h$ , then  $\text{val}_v(h) = \text{val}(h(y))$  for any  $y \in (\mathbb{K}^*)^\Lambda$  with  $\text{val}(y) = v$ .

**Proof:**

Obviously  $\text{val.}(h)$  is piece-wise linear. If  $v$  is in the interior of a top-dimensional cone of the Gröbner fan of  $h$ ,  $\text{t-in}_v(h) = \underline{x}^{\tilde{\omega}}$  is a monomial. Therefore the minimum  $v \cdot \omega$  in the definition of  $\text{val}_v(h)$  is attained for only one term, namely for  $h_{\tilde{\omega}} \cdot \underline{x}^{\tilde{\omega}}$ , and  $\text{val.}(h) : v \mapsto \text{val}_v(h) = v \cdot \tilde{\omega}$  is linear. Furthermore, if  $\text{val}(y) = v$  then the terms of  $h(y)$  of lowest order in  $t$  come from those terms of  $h$  for which  $v \cdot \omega$  is minimal. Thus  $\text{val}(h(y)) = v \cdot \tilde{\omega} = \text{val}_v(h)$ .  $\square$

Note that if  $y$  is not in the interior of a top-dimensional cone of the Gröbner fan of  $h$ , then the terms of lowest order in  $t$  in  $h(y)$  might cancel and we cannot predict the valuation of  $h(y)$ . The generic valuation is, however, the valuation of  $h(y)$  under the assumption that the terms of lowest order in  $t$  do not cancel.

**Definition 2.5**

With the above notation we define the *generic valuation of the  $j$ -invariant at  $u \in \mathbb{R}^{\mathcal{A}_3}$*  as  $\text{val}_u(j) := \text{val}_u(A) - \text{val}_u(\Delta)$ .

**Remark 2.6**

From Lemma 2.4 it follows that

$$\text{val.}(j) : \mathbb{R}^{\mathcal{A}_3} \longrightarrow \mathbb{R} : u \mapsto \text{val}_u(j)$$

is a piece-wise linear function which is linear on intersections  $D \cap D'$  of a top-dimensional cone  $D$  of the Gröbner fan of  $A$  and a top-dimensional cone  $D'$  of the Gröbner fan of  $\Delta$ . For  $u$  in the open interior of  $D \cap D'$ ,  $\text{val}_u(j) = \text{val}(j(f))$  for any  $f = \sum_{(i,j) \in \mathcal{A}_3} a_{ij} x^i y^j$  with  $\text{val}(a_{ij}) = u_{ij}$ .

## 3. TROPICALIZATIONS AND THE CYCLE LENGTH OF A PLANE TROPICAL CUBIC

In this section we will study the tropicalization of plane cubics as well as the tropicalization of the varieties defined by  $A$  or  $\Delta$  in  $(\mathbb{K}^*)^{\mathcal{A}_3}$ . We therefore start again using the general notation 2.1.

**Definition 3.1**

If  $h \in \mathbb{K}[\underline{x}]$  then the *tropicalization of  $V(h)$*  is  $\{p \in \mathbb{K}^\Lambda \mid h(p) = 0\}$ ,

$$\text{Trop}(V(h)) = \overline{\text{val}(V(h) \cap (\mathbb{K}^*)^\Lambda)} \subseteq \mathbb{R}^\Lambda,$$

is the closure of  $\text{val}(V(h) \cap (\mathbb{K}^*)^\Lambda)$  with respect to the Euclidean topology in  $\mathbb{R}^\Lambda$ , where by abuse of notation

$$\text{val} : (\mathbb{K}^*)^\Lambda \longrightarrow \mathbb{Q}^\Lambda : (k_\lambda \mid \lambda \in \Lambda) \mapsto (\text{val}(k_\lambda) \mid \lambda \in \Lambda)$$

denotes the Cartesian product of the valuation map from Section 2.

This definition is not too helpful when it comes down to actually computing tropical varieties. There it is better to consider tropical polynomials.

### Definition 3.2

For a *tropical polynomial*  $F = \min\{u_\omega + v \cdot \omega \mid \omega \in \mathcal{A}\}$ , with  $\mathcal{A} \subset \mathbb{N}^\Lambda$  finite and  $u_\omega \in \mathbb{R}$ , (see e.g. [13]) we define the *tropical hypersurface* associated to  $F$  to be the locus of non-differentiability of the function

$$\mathbb{R}^\Lambda \longrightarrow \mathbb{R} : v \mapsto \min_{\omega \in \mathcal{A}} \{u_\omega + v \cdot \omega\}$$

(i.e. the locus where the minimum is attained by at least two terms). We call the convex hull of  $\mathcal{A}$  the *Newton polytope* of  $F$  respectively of the tropical hypersurface defined by  $F$ . If  $\#\Lambda = 2$  then we call a tropical hypersurface simply a *tropical curve*.

### Remark 3.3

Given  $h = \sum_{\omega \in \mathcal{A}} h_\omega \cdot \underline{x}^\omega \in \mathbb{K}[\underline{x}]$  we define the *tropicalization of  $h$*  to be the tropical polynomial

$$\text{trop}(h) = \min \{ \text{val}(h_\omega) + v \cdot \omega \mid \omega \in \mathcal{A} \}.$$

Then by Kapranov's Theorem (see [2, Theorem 2.1.1]),  $\text{Trop}(V(h))$  is equal to the tropical hypersurface associated to  $\text{trop}(h)$ .

Moreover, this is obviously the collection of  $v$  for which  $\text{t-in}_v(h)$  is not a monomial. In particular, if  $h \in \mathbb{Q}[\underline{x}]$  has constant rational coefficients then it is the codimension-one skeleton of the Gröbner fan.

We will use this definition mainly in two different settings:

- (a) For a plane cubic  $f = \sum_{(i,j) \in \mathcal{A}_3} a_{ij} x^i y^j$  we will consider the corresponding *plane tropical cubic*  $\text{Trop}(V(f)) \subset \mathbb{R}^2$  given by the tropical polynomial  $\min_{(i,j) \in \mathcal{A}_3} \{u_{ij} + ix + jy\}$  where  $\text{val}(a_{ij}) = u_{ij}$ .
- (b) For the numerator and denominator of the  $j$ -invariant  $A$  and  $\Delta$  which are polynomials in  $\mathbb{Q}[\underline{a}]$  we will consider their tropicalizations  $\text{Trop}(V(A))$  and  $\text{Trop}(V(\Delta))$  in  $\mathbb{R}^{\mathcal{A}_3}$ . The latter was recently studied in [1]. As  $A$  and  $\Delta$  have constant rational coefficients these two tropical hypersurfaces are equal to the codimension-one skeletons of the Gröbner fans of  $A$  respectively  $\Delta$ .

Tropical hypersurfaces as defined above are dual to certain marked subdivisions of  $\mathcal{A}$ . Let us recall the necessary facts from [4], still using Notation 2.1.

### Definition 3.4

A *marked polytope* is a pair  $(Q, \mathcal{A})$  where  $Q \subset \mathbb{R}^\Lambda$  is a convex lattice polytope and  $\mathcal{A} \subseteq Q \cap \mathbb{Z}^\Lambda$  contains at least the vertices of  $Q$ . The set  $\mathcal{A}$  is said to be the set of *marked lattice points*.

The Newton polytope  $(Q_3, \mathcal{A}_3)$  as shown in Figure 1 is a marked polytope.

**Definition 3.5**

Let  $(Q, \mathcal{A})$  be a marked polytope in  $\mathbb{R}^\Lambda$  with  $\dim(Q) = \#\Lambda$ . A *marked subdivision* of  $(Q, \mathcal{A})$  is a finite family of marked polytopes  $\{(Q_i, \mathcal{A}_i) \mid i = 1, \dots, k\}$  such that

- (a)  $(Q_i, \mathcal{A}_i)$  is a marked polytope with  $\dim(Q_i) = \#\Lambda$  for  $i = 1, \dots, k$ ,
- (b)  $Q = \bigcup_{i=1}^k Q_i$  is a subdivision of  $Q$ , i.e.  $Q_i \cap Q_j$  is a face (possibly empty) of  $Q_i$  and of  $Q_j$  for all  $i, j = 1, \dots, k$ ,
- (c)  $\mathcal{A}_i \subset \mathcal{A}$  for  $i = 1, \dots, k$ , and
- (d)  $\mathcal{A}_i \cap (Q_i \cap Q_j) = \mathcal{A}_j \cap (Q_i \cap Q_j)$  for all  $i, j = 1, \dots, k$ .

We do not mandate that  $\bigcup_{i=1}^k \mathcal{A}_i = \mathcal{A}$ . Example 3.10 shows an example of a marked subdivision of  $(Q_3, \mathcal{A}_3)$ .

**Definition 3.6**

Let  $S = \{(Q_i, \mathcal{A}_i) \mid i = 1, \dots, k\}$  and  $S' = \{(Q'_j, \mathcal{A}'_j) \mid j = 1, \dots, k'\}$  be subdivisions of  $(Q, \mathcal{A})$ . We say that  $S$  *refines*  $S'$  if for all  $j = 1, \dots, k'$ , the collection of  $(Q_i, \mathcal{A}_i)$  so that  $Q_i \subseteq Q'_j$  is a marked subdivision of  $(Q'_j, \mathcal{A}'_j)$ .

Figure 2 shows two marked subdivisions of  $(Q_3, \mathcal{A}_3)$  where the right one is a refinement of the left one.

**Remark 3.7**

Let us note that the *coarsest subdivision* of  $(Q, \mathcal{A})$  is  $\{(Q, \mathcal{A})\}$ .

More interesting examples of subdivisions are the so called *regular* or *coherent subdivisions*. Given  $\psi \in \mathbb{R}^\Lambda$ , we can associate a subdivision as follows. Let the *upper hull*  $\text{UH}(\psi)$  of  $\psi$  be the convex hull of the subset of  $\mathbb{R}^\Lambda \times \mathbb{R}$  given by

$$S = \{(\omega, a) \mid \omega \in \mathcal{A}, a \geq \psi(\omega)\}.$$

The lower faces of  $\text{UH}(\psi)$  project to  $Q$  giving a subdivision of  $Q$ . Define the *lower convexity*  $\text{LC}(\psi)$  of  $\psi$  to be the function  $\text{LC}(\psi) : Q \rightarrow \mathbb{R}$  whose graph is the lower faces of  $\text{UH}(\psi)$ . This is a convex function. The faces of the subdivision are the maximal domains of linearity of  $\text{LC}(\psi)$ . For a face  $Q_i$ , define the marked set

$$\mathcal{A}_i = \{\omega \in Q_i \cap \mathcal{A} \mid \psi(\omega) = \text{LC}(\psi)(\omega)\},$$

the set of all points of  $\mathcal{A}$  in  $Q_i$  that lie on the lower faces of the upper hull. Marked subdivisions that arise in this fashion are said to be *regular*.

The set of all  $\psi \in \mathbb{R}^\Lambda$  that induce the same regular marked subdivision form an open polyhedral cone in  $\mathbb{R}^\Lambda$ . All such cones form the *secondary fan* of  $\mathcal{A}$ . The secondary fan is the normal fan to a polytope, the *secondary polytope*. The poset of cones in the secondary fan is isomorphic to the poset of regular marked subdivisions under refinement. *Top dimensional cones of the secondary fan* of  $\mathcal{A}$  therefore correspond to the finest subdivisions where each  $Q_i$  is a  $\#\Lambda$ -dimensional simplex and  $\mathcal{A}_i$  is the set of vertices of  $Q_i$ . (See [4, Chap. 7].)

**Remark 3.8**

Note that the minimal cone of the secondary fan is a  $\#\Lambda + 1$ -dimensional space  $L$  and is given by all  $\psi \in \mathbb{R}^\Lambda$  of the form

$$\psi : \mathcal{A} \rightarrow \mathbb{R} : \omega \mapsto a + v \cdot \omega$$

with  $a \in \mathbb{R}$  and  $v \in \mathbb{R}^\Lambda$ . All these functions induce the *coarsest subdivision*. We call  $L$  the *linearity space* of the secondary fan of  $\mathcal{A}_3$ . Every cone of the secondary fan of  $\mathcal{A}_3$  contains its linearity space. Therefore, we may consider the secondary fan as living in  $\mathbb{R}^\Lambda/L$ . When we speak of *rays* in the secondary fan, we mean  $n + 2$ -dimensional cones in the fan containing  $L$ .

The secondary fan of  $\mathcal{A}_3$  is an important object because we will see that the “cycle length” as a function is linear on each top-dimensional cone of the secondary fan. Since we have already seen that the valuation of the  $j$ -invariant is linear on each cone of the common refinement of the Gröbner fans of  $A$  and  $\Delta$  our strategy will be to compare the secondary fan with these two Gröbner fans.

**Remark 3.9**

The *duality* of tropical hypersurfaces in  $\mathbb{R}^\Lambda$  with Newton polytope  $Q_H \subset \mathbb{R}^\Lambda$  and regular marked subdivisions of  $(Q_H, \mathcal{A}_H)$  with  $\mathcal{A}_H = Q_H \cap \mathbb{Z}^\Lambda$  is set up as follows. Given a tropical polynomial  $H = \min_{\omega \in \mathcal{A}_H} \{u_\omega + v \cdot \omega\}$  we associate to  $H$  the regular marked subdivision induced by

$$\psi_H : \mathcal{A}_H \longrightarrow \mathbb{R} : \omega \mapsto u_\omega.$$

This marked subdivision is then dual to the tropical hypersurface defined by  $H$  in the sense of [11, Prop. 3.11], which in its full generality is rather technical. However, for the cases we are interested in it can easily be described.

If  $H$  defines a plane tropical curve  $\mathcal{C}_H$  in  $\mathbb{R}^2$  then each marked polytope of the subdivision of  $(Q_H, \mathcal{A}_H)$  is dual to a vertex of  $\mathcal{C}_H$ , and each facet of a marked polytope is dual to an edge of  $\mathcal{C}_H$ . Moreover, if the facet, say  $F$ , has end points  $(x_1, y_1)$  and  $(x_2, y_2)$  then the *direction vector*  $v(E)$  of the dual edge  $E$  in  $\mathcal{C}_H$  is defined (up to sign) as

$$v(E) = (y_2 - y_1, x_1 - x_2)$$

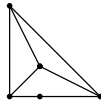
and points in the direction of  $E$ . In particular, the edge  $E$  is orthogonal to its dual facet  $F$ . Finally, the edge  $E$  is unbounded if and only if its dual facet  $F$  is contained in a facet of  $Q_H$ .



The second case we are interested in are the tropical hypersurfaces defined by  $\text{trop}(A)$  and  $\text{trop}(\Delta)$ , i.e. the case when the tropical hypersurface is actually the codimension-one skeleton of a fan. In this case the corresponding fan is just the normal fan of the Newton polytope of the defining polynomial (see [4, Chap. 5]).

**Example 3.10**

The marked subdivision below is for example induced by the tropical polynomial  $\min\{3x, 3y, 0, x, -1 + x + y\}$ .





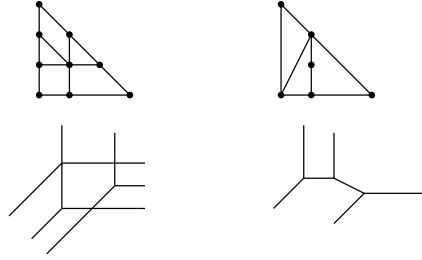
Let us now restrict our attention to plane tropical cubics, that is, curves defined by tropical polynomials whose Newton polytope is *contained* in the triangle  $Q_3$ . Note that this triangle has only one interior point, so also all possible marked subdivisions we consider have at most one interior point.

**Definition 3.11**

We say that a plane tropical cubic  $\mathcal{C}$  *has a cycle* if the interior point  $(1, 1)$  is visible as the vertex of a marked polytope in its dual marked subdivision. If this is the case, the *cycle* of  $\mathcal{C}$  is the union of those bounded edges of  $\mathcal{C}$  which are dual to the facets of marked polytopes in the marked subdivision which emanate from  $(1, 1)$ , and we say that these edges *form the cycle*.

**Example 3.12**

In the picture below, the left plane tropical cubic has a cycle while the right one does not, since  $(1, 1)$  is visible but it is not a vertex of one of the marked polytopes in the subdivision.



**Definition 3.13**

For a bounded edge  $E$  of a plane tropical curve with direction vector  $v(E)$ , defined as in Remark 3.9 (i.e.  $v(E)$  is orthogonal to its dual facet in the marked subdivision and of the same Euclidean length as this facet) we define the *lattice length*  $l(E) = \frac{\|E\|}{\|v(E)\|}$  to be the Euclidean length of  $E$  divided by the Euclidean length of  $v(E)$ .

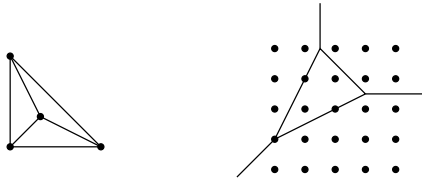
For a plane tropical cubic with cycle, we define its *cycle length* to be the sum of the lattice lengths of the edges which form the cycle. If the plane tropical cubic has no cycle we say its length is zero. This defines a function “cycle length”

$$\text{cl} : \mathbb{R}^{\mathcal{A}_3} \rightarrow \mathbb{R} : u = (u_\omega \mid \omega \in \mathcal{A}_3) \mapsto \text{cl}(u) = \text{“cycle length of } \mathcal{C}_H\text{”}$$

associating to every plane tropical cubic polynomial  $H = \min_{\omega \in \mathcal{A}_3} \{u_\omega + v \cdot \omega\}$  the cycle length of the corresponding plane tropical cubic  $\mathcal{C}_H$ .

**Example 3.14**

The following picture shows a plane tropical cubic with cycle length  $\frac{9}{2}$ .



**Definition 3.15**

Assume a plane tropical cubic has no cycle, i.e.  $(1, 1)$  is not the vertex of a marked polytope in the corresponding marked subdivision, but it is contained in a facet of (necessarily) two such marked polytopes, say  $(Q_1, \mathcal{A}_1)$  and  $(Q_2, \mathcal{A}_2)$ . They are dual to two vertices  $V_1$  and  $V_2$  of the tropical curve. We define the *generalized cycle length* of such a cubic to be four times the lattice length of the edge connecting  $V_1$  and  $V_2$ .

This definition is necessary because these tropical curves arise as limits of curves with a cycle, and the cycle length tends to the generalized cycle length for such a limit: One factor of 2 is necessary because two edges tend to the same edge in the

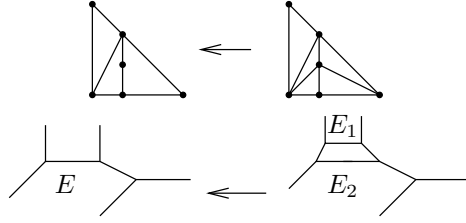


FIGURE 2. Generalized cycle length

picture. The other factor of 2 appears because the direction vector of  $E$  is of twice the euclidean length than the direction vector of  $E_1$  and  $E_2$ .

Let us generalize Definition 3.11 to plane tropical curves other than cubics.

**Definition 3.16**

Let  $\mathcal{C}$  be a plane tropical curve with Newton polytope  $Q$  and with dual marked subdivision  $\{(Q_i, \mathcal{A}_i) \mid i = 1, \dots, l\}$ . Suppose that  $\tilde{\omega} \in \text{Int}(Q) \cap \mathbb{Z}^2$  is in the interior of  $Q$  and that the  $(Q_i, \mathcal{A}_i)$  are ordered such that  $\tilde{\omega}$  is a vertex of  $Q_i$  for  $i = 1, \dots, k$  and it is not contained in  $Q_i$  for  $i = k+1, \dots, l$  (see Figure 3). We then say that

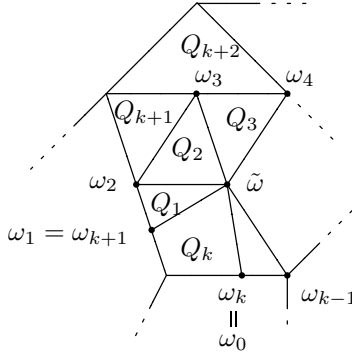


FIGURE 3. Marked subdivision determining a cycle

$\tilde{\omega}$  determines a cycle of  $\mathcal{C}$ , namely the union of the edges of  $\mathcal{C}$  dual to the facets emanating from  $\tilde{\omega}$ , and we say that these edges form the cycle determined by  $\tilde{\omega}$ . The length of this cycle is defined as in Definition 3.13.

**Lemma 3.17**

Let  $(Q, \mathcal{A})$  be a marked polytope in  $\mathbb{R}^2$  with a regular marked subdivision  $\{(Q_i, \mathcal{A}_i) \mid i = 1, \dots, l\}$  and suppose that  $\tilde{\omega} \in \text{Int}(Q) \cap \mathbb{Z}^2$  in the interior of  $Q$  is a vertex of  $Q_i$  for  $i = 1, \dots, k$  and it is not contained in  $Q_i$  for  $i = k+1, \dots, l$ .

If  $\psi : \mathcal{A} \rightarrow \mathbb{R}$  is a function defining the subdivision, then  $\tilde{\omega}$  determines a cycle in the plane tropical curve  $\mathcal{C}$  given by the tropical polynomial

$$\min\{\psi(\omega) + v \cdot \omega \mid \omega \in \mathcal{A}\}$$

and, using the notation in Figure 3, its length is

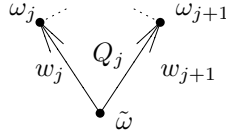
$$\sum_{j=1}^k (\psi(\tilde{\omega}) - \psi(\omega_j)) \cdot \frac{D_{j-1,j} + D_{j,j+1} + D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}}$$

where  $D_{i,j} = \det(w_i, w_j)$  with  $w_i = \omega_i - \tilde{\omega}$  and  $w_j = \omega_j - \tilde{\omega}$ .

**Proof:**

By definition  $\tilde{\omega}$  determines a cycle. It remains to prove the statement on its length.

For this we consider the convex polytope  $Q_j$  having  $\omega_{j+1}$ ,  $\tilde{\omega}$  and  $\omega_j$  as neighboring vertices:



The vertex  $v_j = (v_{j,1}, v_{j,2})$  of  $\mathcal{C}$  dual to  $Q_j$  is given by the system of linear equations

$$\omega_j \cdot v_j + u_j = \omega_{j+1} \cdot v_j + u_{j+1} = \tilde{\omega} \cdot v_j + u,$$

where  $u_j = \psi(\omega_j)$ ,  $u_{j+1} = \psi(\omega_{j+1})$  and  $u = \psi(\tilde{\omega})$ . This system can be rewritten as

$$\begin{pmatrix} w_j^t \\ w_{j+1}^t \end{pmatrix} \cdot v_j = \begin{pmatrix} u - u_j \\ u - u_{j+1} \end{pmatrix}.$$

Since  $\omega_{j+1}$ ,  $\tilde{\omega}$  and  $\omega_j$  are neighboring vertices of the polytope  $Q_j$  the vectors  $w_j$  and  $w_{j+1}$  are linearly independent and we may apply Cramer's Rule to find

$$v_{j,1} = \frac{\det \begin{pmatrix} u - u_j & w_{j,2} \\ u - u_{j+1} & w_{j+1,2} \end{pmatrix}}{D_{j,j+1}} \quad \text{and} \quad v_{j,2} = \frac{\det \begin{pmatrix} w_{j,1} & u - u_j \\ w_{j+1,1} & u - u_{j+1} \end{pmatrix}}{D_{j,j+1}}. \quad (1)$$

The lattice length of the edge from  $v_{j-1}$  to  $v_j$  is the real number  $\lambda_j \in \mathbb{R}$  such that  $(v_j - v_{j-1}) = \lambda_j \cdot w_j^\perp$ , where  $w_j^\perp = (-w_{j,2}, w_{j,1})$  is perpendicular to  $w_j$ . Thus

$$\lambda_j = \frac{(v_j - v_{j-1}) \cdot w_j^\perp}{w_j^\perp \cdot w_j^\perp} = \frac{(v_j - v_{j-1}) \cdot w_j^\perp}{w_j \cdot w_j}. \quad (2)$$

In order to understand the right hand side of this equation better we need the following observation. The last row of the matrix

$$M = \begin{pmatrix} w_{j-1,1} & w_{j,1} & w_{j+1,1} \\ w_{j-1,2} & w_{j,2} & w_{j+1,2} \\ w_{j-1} \cdot w_j & w_j \cdot w_j & w_{j+1} \cdot w_j \end{pmatrix}$$

is a linear combination of the first two, and thus the determinant of  $M$  is zero. Developing the determinant by the last row we get

$$0 = \det(M) = w_{j-1} \cdot w_j \cdot D_{j,j+1} - w_j \cdot w_j \cdot D_{j-1,j+1} + w_{j+1} \cdot w_j \cdot D_{j-1,j},$$

or equivalently

$$\frac{D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}} = -\frac{D_{j-1,j+1}}{D_{j-1,j} \cdot D_{j,j+1}} = -\frac{w_{j-1} \cdot w_j}{w_j \cdot w_j \cdot D_{j-1,j}} - \frac{w_{j+1} \cdot w_j}{w_j \cdot w_j \cdot D_{j,j+1}}.$$

Expanding the right hand side of (2) using (1) and plugging in this last equality we get

$$\begin{aligned} \lambda_j &= \frac{u - u_{j-1}}{D_{j-1,j}} + \frac{u - u_{j+1}}{D_{j,j+1}} - (u - u_j) \cdot \left( \frac{w_j \cdot w_{j+1}}{w_j \cdot w_j \cdot D_{j,j+1}} + \frac{w_{j-1} \cdot w_j}{w_j \cdot w_j \cdot D_{j-1,j}} \right) \\ &= \frac{u - u_{j-1}}{D_{j-1,j}} + \frac{u - u_{j+1}}{D_{j,j+1}} + \frac{(u - u_j) \cdot D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}}. \end{aligned}$$

The lattice length of the cycle of  $\mathcal{C}$  is then given by adding the  $\lambda_j$ , i.e. it is

$$\begin{aligned} \lambda_1 + \dots + \lambda_k &= \sum_{j=1}^k \frac{u - u_{j-1}}{D_{j-1,j}} + \frac{u - u_{j+1}}{D_{j,j+1}} + \frac{(u - u_j) \cdot D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}} \\ &= \sum_{j=1}^k (u - u_j) \cdot \left( \frac{D_{j-1,j} + D_{j,j+1} + D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}} \right). \end{aligned}$$

□

### Remark 3.18

An immediate consequence of Lemma 3.17 is that the function “cycle length”,  $\text{cl}$ , from Definition 3.13 is linear on each cone of the secondary fan of  $\mathcal{A}_3$ .

## 4. THE MAIN THEOREM

### Theorem 4.1

Let  $\mathcal{C}$  be a plane tropical cubic given by the tropical polynomial

$$\min_{(i,j) \in \mathcal{A}_3} \{u_{ij} + ix + jy\}$$

and assume that  $\mathcal{C}$  has a cycle.

Then the negative of the generic valuation of the  $j$ -invariant at  $u = (u_{ij})_{(i,j) \in \mathcal{A}_3}$  is equal to the cycle length of  $\mathcal{C}$ , i.e.

$$-\text{val}_u(j) = \text{cl}(u).$$

Furthermore, if the marked subdivision dual to  $\mathcal{C}$  corresponds to a top-dimensional cone of the secondary fan of  $\mathcal{A}_3$  (that is, if it is a triangulation), then  $\text{val}_u(j) = \text{val}(j(f))$  where  $f = \sum_{(i,j) \in \mathcal{A}_3} a_{ij} x^i y^j$  is any elliptic curve over  $\mathbb{K}$  with coefficients  $a_{ij}$  satisfying  $\text{val}(a_{ij}) = u_{ij}$ .

There are two main parts of the proof: the first part is to compare certain “domains of linearity” in  $\mathbb{R}^{\mathcal{A}_3}$  of the two piece-wise linear functions “cycle length”,  $\text{cl}$ , and “generic valuation of  $j$ ”,  $\text{val}(j)$ , and the second part is to compare the two linear functions on each domain.

The proof uses many results that will be proved in the following sections.

**Proof of Theorem 4.1:**

Note that our claim only involves curves  $\mathcal{C}$  which have a cycle or, equivalently, where in the dual subdivision the point  $(1, 1)$  is a vertex of a marked polytope. Therefore we may replace  $\mathbb{R}^{\mathcal{A}_3}$  as domain of definition of  $\text{cl}$  and  $\text{val}_u(j)$  by the union  $U$  of those cones of the secondary fan of  $\mathcal{A}_3$  where the corresponding marked subdivision contains  $(1, 1)$  as a vertex of a marked polytope. The coordinates on  $U$  are given by  $u = (u_{ij} \mid (i, j) \in \mathcal{A}_3)$  and the canonical basis vector  $e_{kl} = (\delta_{ik} \cdot \delta_{jl} \mid (i, j) \in \mathcal{A}_3)$  has a one in position  $kl$  and zeros elsewhere.

From Lemma 7.2 we know that  $U$  is contained in a single cone of the Gröbner fan of  $A$ , namely the one dual to the vertex  $12 \cdot e_{11}$  of the Newton polytope of  $A$ . Hence the generic valuation of  $A$  is linear on  $U$ , namely

$$U \rightarrow \mathbb{R} : u \mapsto \text{val}_u(A) = 12 \cdot u_{11}.$$

Thus, if we want to divide  $U$  into cones on which  $\text{val}_u(j)$  is linear, it suffices to consider  $u \mapsto \text{val}_u(\Delta)$ , and we know already that the latter is linear on cones of the Gröbner fan of  $\Delta$  by Lemma 2.4. Thus so is  $\text{val}_u(j)$  restricted to  $U$ , and by Lemma 5.2 and Remark 5.1 the function  $\text{cl}$  is so as well. Moreover, Remark 5.1 tells us that  $U$  is indeed a union of cones of the Gröbner fan of  $\Delta$ , and each such cone is a union of certain  $\Delta$ -equivalent cones of the secondary fan of  $\mathcal{A}_3$ .

Hence to prove that the two functions  $\text{val}_u(j)$  and  $\text{cl}$  coincide it is enough to compare the linear functions on each cone of the Gröbner fan of  $\Delta$  contained in  $U$ . To do this, we use Theorem 11.3.2 of [4] which enables us to compute the assignment rule for the linear function  $u \mapsto \text{val}_u(\Delta)$  on each such cone, say  $D$ , given a (top-dimensional) marked subdivision  $T$  whose corresponding cone in the secondary fan of  $\mathcal{A}_3$  is contained in  $D$ . In fact, it provides us with a formula to compute the coefficient of  $u_{ij}$  for each  $(i, j) \in \mathcal{A}_3$ . Since we already know that the two functions  $u \mapsto \text{val}_u(\Delta)$  and  $\text{cl}$  are linear on  $D$ , we can for our comparison assume that  $T$  is the representative of its class with as few edges as possible. The coefficient of  $u_{ij}$  in the linear function  $\text{cl}$  for the marked subdivision is given by Lemma 3.17. To compare the two coefficients, there are some cases to distinguish, which is done by Lemma 5.5. This proves the first part of the theorem.

Finally, for any point  $u$  in the interior of a cone of the Gröbner fan of  $\Delta$ ,  $\text{val}_u(j) = \text{val}(j(f))$  for any polynomial  $f = \sum_{(i,j) \in \mathcal{A}_3} a_{ij} x^i y^j$  with  $\text{val}(a_{ij}) = u_{ij}$  by Lemma 2.4. As a point  $u$  in the interior of a top-dimensional cone of the secondary fan of  $\mathcal{A}_3$  is in the interior of a cone of the Gröbner fan of  $\Delta$ , the last statement follows as well.  $\square$

We would like to give an alternative proof of the statement whose methods we believe to be interesting on their own. Here, we will consider smaller domains of linearity, namely the cones of the secondary fan of  $\mathcal{A}_3$  contained in  $U$  (using the notation from the above proof).

**Alternative proof of Theorem 4.1:**

From [4, Chap. 10, Thm. 1.2] or alternatively from Lemma 6.5 we conclude that the codimension 1 cones of the Gröbner fan of  $\Delta$  do not meet the interior of any top-dimensional cone of the secondary fan of  $\mathcal{A}_3$ . Thus an open top-dimensional

cone of the secondary fan of  $\mathcal{A}_3$  is completely contained in some top-dimensional cone of the Gröbner fan of  $\Delta$ , and using Lemma 2.4 we conclude that  $u \mapsto \text{val}_u(\Delta)$  is linear on each top-dimensional cone of the secondary fan of  $\mathcal{A}_3$ . Using Lemma 7.2 we can see that  $u \mapsto \text{val}_u(A)$  is linear on each cone of the secondary fan of  $\mathcal{A}_3$  corresponding to a subdivision for which the interior point is visible. By 3.17 we know that the function  $\text{cl}$  is linear on a cone of the secondary fan of  $\mathcal{A}_3$ , too. To show that the two functions agree, we only have to show that they agree on the rays of each cone of the secondary fan of  $\mathcal{A}_3$  in question. In Proposition 8.4 we classify the rays. Then a computation for each ray shows that the two functions agree. We computed this using the procedure `raysC` in the library `jinvariant.lib` available via the URL

<http://www.mathematik.uni-kl.de/~keilen/en/jinvariant.html>.

Note that we have to compare with the generalized cycle length, because a point on a ray is a limit of points corresponding to curves with a cycle. Since the rays of the secondary fan of  $\mathcal{A}_3$  in question are not necessarily contained in the interior of a top-dimensional cone of the Gröbner fan of  $A$  and  $\Delta$ , we have to use the generic valuation. However, for a point  $u$  in the interior of a top-dimensional cone of the secondary fan of  $\mathcal{A}_3$  we know that  $\text{val}_u(j) = \text{val}(j(f))$  for any polynomial  $f = \sum_{(i,j) \in \mathcal{A}_3} a_{ij} x^i y^j$  with  $\text{val}(a_{ij}) = u_{ij}$  by Lemma 2.4.  $\square$

### Corollary 4.2

*Let  $f = \sum_{(i,j) \in \mathcal{A}_3} a_{ij} x^i y^j$  define a smooth elliptic curve over  $\mathbb{K}$  such that the valuation of its  $j$ -invariant is positive. Then its tropicalization does not have a cycle.*

### Remark 4.3

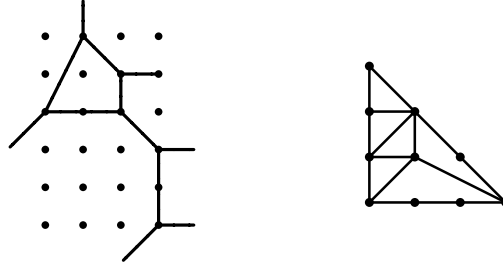
Let  $C$  be a smooth elliptic curve over  $\mathbb{K}$ . It is obvious that the tropicalization of  $C$  depends on the embedding into the projective plane that we choose. One might ask if the cycle length does not depend on the embedding though, as it should take the role of a tropical  $j$ -invariant. This is not true however. For example, each curve can be put into Weierstrass-form without  $xy$ -term and its tropicalization does not have a cycle, because the interior point is not part of the marked subdivision. Also, an embedding might be such that the valuations of the coefficients lie in a cone of the Gröbner fan of  $\Delta$  of higher codimension, and the coefficients might be such that the lowest order terms of  $\Delta(f)$  cancel. But as the cycle length is equal to the generic valuation of  $j$  and the generic valuation of  $j$  is smaller than the valuation of the  $j$ -invariant in this case, the cycle length would not reflect the  $j$ -invariant.

As an example, we consider the following family of curves over  $\mathbb{K}$  with the same  $j$ -invariant (hence you could also say: a family of different embeddings for one curve). We choose a given curve and apply the coordinate change (in affine coordinates)  $(x, y) \mapsto (x + k, y)$ , where  $k \in \mathbb{K}$ . Since this is an isomorphism, the curves over  $\mathbb{K}$  have the same  $j$ -invariant for any choice of  $k$ . In particular, the valuation of the  $j$ -invariant is the same for any  $k$ . One might hope that at least for a general choice of  $k$  the cycle length of the tropicalization is equal to the valuation of the  $j$ -invariant. But even this is not true, as the following example demonstrates. Let us take a subset of the family given by the coordinate changes  $(x, y) \mapsto (x + t^b, y)$ , where  $b \in \mathbb{Q}$ . For our example we will see that infinitely many choices of  $b$  lead

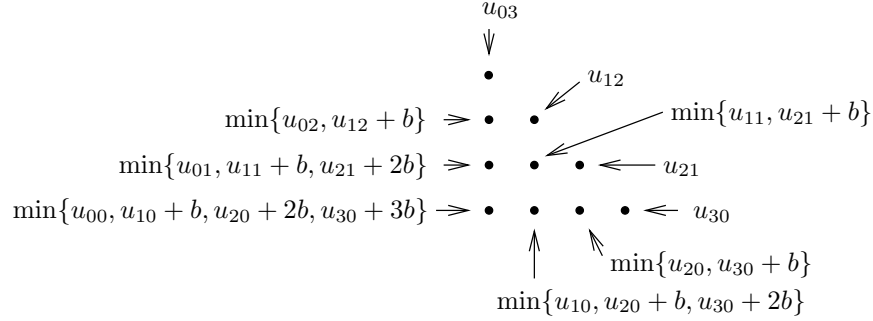
to the expected cycle length, but also infinitely many lead to the “wrong” cycle length. This shows that it is not true that a general choice of  $k$  (in the sense of Zariski topology) leads to the expected cycle length. As example we choose

$$f = c_{00} \cdot t + c_{10} \cdot t^{100} \cdot x + c_{20} \cdot t^{100} \cdot x^2 + c_{30} \cdot t \cdot x^3 + c_{01} \cdot t \cdot y \\ + c_{11} \cdot x \cdot y + c_{21} \cdot t^{100} \cdot x^2 \cdot y + c_{02} \cdot t^3 \cdot y^2 + c_{12} \cdot t \cdot x \cdot y^2 + c_{03} \cdot t^7 \cdot y^3,$$

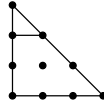
where the  $c_{ij} \in \mathbb{C}$  are general. By general we mean that after applying the coordinate change every coefficient has the expected valuation and nothing cancels. The valuation of the  $j$ -invariant is 5. The cycle length of the tropicalization is 5, too, as expected.



Let us check what happens to the valuations  $u_{ij}$  of the coefficients  $c_{ij}t^{u_{ij}}$  when we apply the coordinate change. In general (i.e. if no cancellation happens) the valuations are as follows:

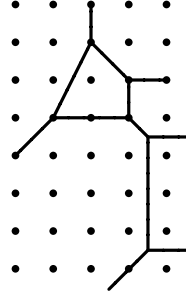


If we choose  $b$  very small, then all points but the point  $(0, 3)$  lie on the same plane and thus the subdivision is as follows:

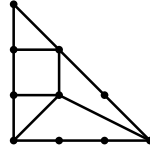


In fact, it corresponds to a ray of the secondary fan of  $\mathcal{A}_3$ .

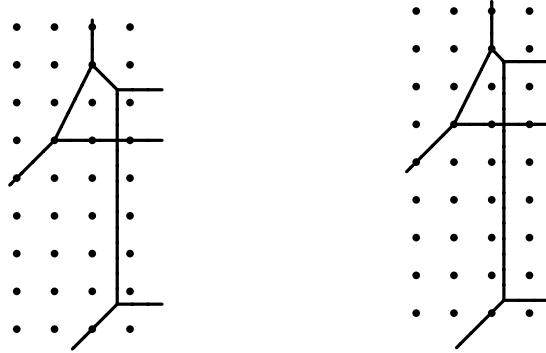
For our special choice of  $f$  and the coefficients  $u_{ij}$  as above, this marked subdivision is only reached when  $b \leq -1$ . Starting from the other end we see that the tropical curve stays unchanged as long as  $b \geq 2$ . In particular, the cycle length is as expected for all those  $b$ . For  $b$  in the interval  $1 < b \leq 2$  the dual marked subdivision is not changed, but the position of two vertices of the tropical curve changes. For example for  $b = \frac{3}{2}$ , the tropical curve is:



For  $b = 1$  the marked subdivision changes to



and it remains like this for  $0 \leq b \leq 1$ , while the cycle length decreases; e.g. for  $b = \frac{2}{3}$  respectively  $b = \frac{1}{3}$  we get:



This happens because the cycle length is equal to the generic valuation of  $j$ . The latter is not equal to the valuation of the  $j$ -invariant if and only if the  $t$ -initial form  $t\text{-in}_u(\Delta)$  cancels when plugging in the leading terms of the Puiseux series coefficients. The  $t$ -initial form of  $\Delta$  corresponding to this marked subdivision is  $a_{11}^7 a_{00} a_{30} a_{01} a_{12}^2 - a_{11}^8 a_{00} a_{30} a_{12} a_{02} = a_{11}^7 a_{00} a_{30} a_{12} \cdot (a_{01} a_{12} - a_{11} a_{02})$ . The leading term of the coefficient for  $f(x + t^b, y)$  of  $y$  is  $c_{11} t^b$ , the leading term for  $xy^2$  is  $c_{12} t$ , the one for  $xy$  is  $c_{11}$ , and the one for  $y^2$  is  $c_{12} t^{1+b}$ . Plugging those into  $a_{01} a_{12} - a_{11} a_{02}$  yields  $c_{11} t^b \cdot c_{12} t - c_{11} \cdot c_{12} t^{1+b} = 0$ . Thus the generic valuation of  $j$  (whose negative is equal to the decreasing cycle length) is not equal to the valuation of the  $j$ -invariant for values of  $b$  in a whole interval.

We computed this example (as well as many other examples) using the procedure `drawtropicalcurve` from the SINGULAR library `tropical.lib` (see [7]) which can be obtained via the URL

<http://www.mathematik.uni-kl.de/~keilen/en/tropical.html>.

This library contains also a procedure `tropicalJInvariant` which computes the cycle length of a tropical curve as defined in Definition 3.13.



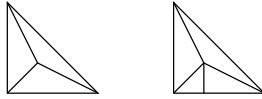
5.  $\Delta$ -EQUIVALENT MARKED SUBDIVISIONS

In this section we want to show that the function “cycle length”,  $\text{cl}$ , is linear on the union of cones of the secondary fan of  $\mathcal{A}_3$  which are  $\Delta$ -equivalent. Also, we provide the classification of the different cases we need to consider in order to compare the two linear functions  $\text{val}(j)$  and  $\text{cl}$  on such a union. This is part of our first proof of Theorem 4.1.

**Remark 5.1**

The Prime Factorization Theorem, [4, Chap. 10, Thm. 1.2], or alternatively Lemma 6.5 tells us that the codimension 1 cones of the Gröbner fan of  $\Delta$  do not meet the interior of any top-dimensional cone of the secondary fan of  $\mathcal{A}_3$ . Thus the Gröbner fan of  $\Delta$  is a coarsening of the secondary fan of  $\mathcal{A}_3$ . Two cones of the secondary fan of  $\mathcal{A}_3$  are called  $\Delta$ -equivalent if they are contained in the same cone of the Gröbner fan of  $\Delta$ .

It has been studied how two top-dimensional marked subdivisions whose cones belong to the same  $\Delta$ -equivalence class can differ. By [4, Chap. 11, Prop. 3.8] they can be obtained from each other by a sequence of modifications along a circuit (see [4, Chap. 7, Sect. 2C]) such that each intermediate (top-dimensional) marked subdivision belongs to the same equivalence class. Since our point configuration (i.e.  $\mathcal{A}_3$ , the integer points of the triangle  $Q_3$ ) is in the plane, we can use [4, Chap. 11, Prop. 3.9] to see that if a marked subdivision can be obtained from another equivalent one by a modification along a circuit, then this circuit consists of three collinear points on the boundary of  $Q_3$ . An example is shown in the following picture, the three points are  $(0, 0)$ ,  $(1, 0)$  and  $(3, 0)$ .

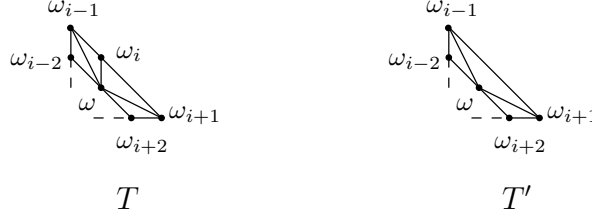
**Lemma 5.2**

*The function  $\text{cl}$  (see Definition 3.13) is linear on a union of cones of the secondary fan of  $\mathcal{A}_3$  which are  $\Delta$ -equivalent (i.e. on a cone of the Gröbner fan of  $\Delta$ ).*

**Proof:**

Given two  $\Delta$ -equivalent marked subdivisions  $T$  and  $T'$  of the secondary fan of  $\mathcal{A}_3$ , we can use Lemma 3.17 to determine the function  $\text{cl}$  on the cone corresponding to each of them. Recall from Remark 3.18 that the function is linear on each cone of the secondary fan of  $\mathcal{A}_3$ . We want to show that the assignment rules of these two linear functions coincide.

Without restriction we can assume that  $T$  can be obtained from  $T'$  by a modification along a circuit, and this circuit consists then of three collinear points on a facet of  $Q_3$  (see Remark 5.1).



Recall from Lemma 3.17 that the coefficients of the linear function  $\text{cl}$  can be determined using the determinants  $D_{i,j} = \det(w_i, w_j)$ , where  $w_i = \omega_i - \tilde{\omega}$ . One easily sees that for  $T$  and  $T'$  the following two equations hold:

$$D_{i-1,i} + D_{i,i+1} = D_{i-1,i+1}, \text{ and} \quad (3)$$

$$D_{i,i+1} = \lambda \cdot D_{i-1,i} \text{ for } \lambda \text{ satisfying } \lambda \cdot (w_{i-1} - w_i) = w_i - w_{i+1}. \quad (4)$$

To show that the two assignment rules of  $\text{cl}$  on the cones for  $T$  respectively  $T'$  coincide we have to show that for  $T$  the summand for  $\omega_i$  equals 0 and the summand for  $\omega_{i-1}$  equals the summand for  $\omega_{i-1}$  for  $T'$ . The first statement follows immediately from Equation (3) above. To show the second statement, we subtract the two summands from each other:

$$\frac{D_{i-2,i-1} + D_{i-1,i} + D_{i,i-2}}{D_{i-2,i-1} \cdot D_{i-1,i}} - \frac{D_{i-2,i-1} + D_{i-1,i+1} + D_{i+1,i-2}}{D_{i-2,i-1} \cdot D_{i-1,i+1}}$$

Multiplied with  $(D_{i-1,i} \cdot D_{i-1,i+1})$  this difference is equal to:

$$\begin{aligned} & D_{i-2,i-1} \cdot D_{i-1,i+1} + D_{i-1,i} \cdot D_{i-1,i+1} + D_{i,i-2} \cdot D_{i-1,i+1} \\ & - D_{i-2,i-1} \cdot D_{i-1,i} - D_{i-1,i+1} \cdot D_{i-1,i} - D_{i+1,i-2} \cdot D_{i-1,i} \\ & = D_{i-2,i-1} \cdot D_{i,i+1} + D_{i,i-2} \cdot D_{i,i+1} + D_{i,i-2} \cdot D_{i-1,i} - D_{i+1,i-2} \cdot D_{i-1,i} \\ & = -\det(w_{i-1} - w_i, w_{i-2}) \cdot D_{i,i+1} + \det(w_i - w_{i+1}, w_{i-2}) \cdot D_{i-1,i} = 0 \end{aligned}$$

where the first equality follows from Equation (3) above and the last from (4).  $\square$

### Definition 5.3

Let us fix a cone  $C_T$  of the secondary fan of  $\mathcal{A}_3$  corresponding to the marked subdivision  $T$ . We then denote by  $\eta_T(i, j)$  the coefficient of  $u_{ij}$  in the assignment rule of the linear function  $u \mapsto \text{val}_u(\Delta)$  on  $C_T$ , and by  $c_T(i, j)$  we denote the coefficient of  $u_{ij}$  in the assignment rule of the linear function  $\text{cl}$  restricted to  $C_T$ .

### Remark 5.4

Note that by Lemma 2.4 and Remark 5.1  $\eta_T(i, j) = \eta_{T'}(i, j)$  for all  $(i, j) \in \mathcal{A}_3$  whenever  $T$  and  $T'$  belong to  $\Delta$ -equivalent cones of the secondary fan of  $\mathcal{A}_3$ , and by Lemma 5.2 also  $c_T(i, j) = c_{T'}(i, j)$  for all  $(i, j) \in \mathcal{A}_3$  in this situation.

### Lemma 5.5

Let  $T$  be a marked subdivision of  $(Q_3, \mathcal{A}_3)$  corresponding to a top-dimensional cone in the secondary fan of  $\mathcal{A}_3$  (i.e. a triangulation) such that  $(1, 1)$  is a vertex of some marked polytope in  $T$  (i.e. all dual plane tropical curves have a cycle). Then  $c_T(1, 1) = \eta_T(1, 1) - 12$  and  $c_T(i, j) = \eta_T(i, j)$  for all  $(i, j) \neq (1, 1)$ .

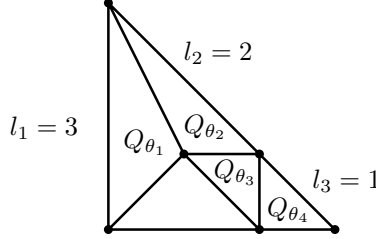
**Proof:**

Due to Remark 5.4 we may for the proof assume that  $T = \{(Q_\theta, \mathcal{A}_\theta) \mid \theta \in \Theta\}$  is the representative of its  $\Delta$ -equivalence class with as few edges as possible.

Moreover, if two triangulations  $T$  and  $T'$  can be transformed into each other by an integral unimodular linear isomorphism, i.e. by linear coordinate change of the projective coordinates  $(x, y, z)$  with a matrix in  $\text{Gl}_3(\mathbb{Z})$ , and the claim holds for  $T$  then it obviously also holds for  $T'$ . In this situation we say that  $T$  and  $T'$  are *symmetric* to each other. We therefore only have to prove the claim *up to symmetry*.

We want to use [4, Chap. 11, Thm. 3.2] which explains how  $\eta_T(i, j)$  can be computed. For each  $(i, j) \in \mathcal{A}_3$  we have to consider all  $(Q_\theta, \mathcal{A}_\theta)$  such that  $(i, j) \in \mathcal{A}_\theta$ . Note that since  $T$  by assumption is a triangulation then  $(i, j) \in \mathcal{A}_\theta$  implies necessarily that  $(i, j)$  is a vertex of  $Q_\theta$ . We have to distinguish four cases, where in the formulas  $\text{vol}(Q_\theta)$  denotes the generalized lattice volume (i.e. twice the euclidean area of  $Q_\theta$ ):

- If  $(i, j)$  is a vertex of  $Q_3$ , then  $\eta_T(i, j) = 1 - l_1 - l_2 + \sum_{(i, j) \in \mathcal{A}_\theta} \text{vol}(Q_\theta)$  where  $l_1$  and  $l_2$  denote the lattice lengths of those facets of some  $Q_\theta$  adjacent to  $(i, j)$  which are contained in facets of  $Q_3$ . E.g. if  $(i, j) = (0, 3)$  in the following triangulation  $T$ , then  $\eta_T(0, 3) = 1 - l_1 - l_2 + \text{vol}(Q_{\theta_1}) + \text{vol}(Q_{\theta_2}) = 1 - 3 - 2 + 3 + 2 = 1$ .



- If  $(i, j)$  lies on a facet of  $Q_3$ , is not a vertex of  $Q_3$ , but is a vertex of some  $Q_{\theta'}$ , then  $\eta_T(i, j) = -l_1 - l_2 + \sum_{(i, j) \in \mathcal{A}_\theta} \text{vol}(Q_\theta)$  where again  $l_1$  and  $l_2$  denote the lattice lengths of those facets of some  $Q_\theta$  adjacent to  $(i, j)$  which are contained in facets of  $Q_3$ , e.g. if in the previous example  $(i, j) = (2, 1)$  then  $\eta_T(i, j) = -l_2 - l_3 + \text{vol}(Q_{\theta_2}) + \text{vol}(Q_{\theta_3}) + \text{vol}(Q_{\theta_4}) = -2 - 1 + 2 + 1 + 1 = 1$ .
- If  $(i, j)$  lies on a facet of  $Q_3$ , is not a vertex of any  $Q_\theta$ , then  $\eta_T(i, j) = 0$ .
- And finally  $\eta_T(1, 1) = \sum_{(1, 1) \in \mathcal{A}_\theta} \text{vol}(Q_\theta)$ .

Let  $Q$  be the union of all those  $Q_\theta$  which contain  $(1, 1)$ , and endow the marked polytope  $(Q, Q \cap \mathcal{A}_3)$  with the subdivision, say  $T_Q$ , induced by  $T$ . We say that  $Q$  meets a facet of  $Q_3$  if the intersection of  $Q$  with this facet is 1-dimensional (and not only a vertex). Moreover, we say that a facet of  $Q$  is multiple if it contains more than two lattice points.

We first want to show that  $\eta_T(i, j)$  and  $c_T(i, j)$  are as claimed whenever  $(i, j) \in Q$ . Up to symmetry, we have to distinguish the following cases for  $Q$  and  $T_Q$ :

- Assume  $Q$  meets all three facets of  $Q_3$ , and assume that for all three facets the intersection with  $Q$  is multiple. Then  $Q$  looks (up to symmetry) like one of the following two pictures:

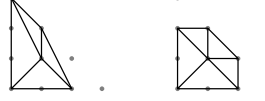


In the second case,  $\eta_T(1, 1) = 8$ . Using Lemma 3.17 we can compute  $c_T(1, 1)$ . It is a sum with a summand for each vertex of  $Q$ . The summand for  $(0, 0)$  is

$$\frac{\det \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} + \det \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} + \det \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}}{\det \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \cdot \det \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}} = -1.$$

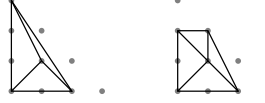
Computing the other 3 summands analogously we get  $c_T(1, 1) = -4 = \eta_T(1, 1) - 12$ . In the first case,  $\eta_T(1, 1) = 9$  and  $c_T(1, 1) = -3$ .

- Assume  $Q$  meets two facets of  $Q_3$  multiply and one facet non-multiply.



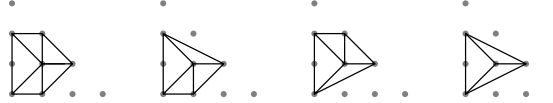
In both cases,  $\eta_T(1, 1) = 7$  and  $c_T(1, 1) = -5$ .

- Assume  $Q$  meets two facets of  $Q_3$  multiply and the third facet not at all.



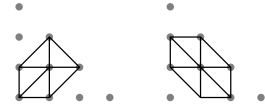
In both cases,  $\eta_T(1, 1) = 6$  and  $c_T(1, 1) = -6$ .

- Assume  $Q$  meets only one facet of  $Q_3$  multiply (and the two remaining facets non-multiply, or only one of them and that one non-multiply, or none of them at all).



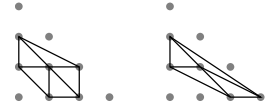
In the first case,  $\eta_T(1, 1) = 6$  and  $c_T(1, 1) = -6$ , in the second and third case,  $\eta_T(1, 1) = 5$  and  $c_T(1, 1) = -7$ , and in the last case,  $\eta_T(1, 1) = 4$  and  $c_T(1, 1) = -8$ .

- Assume  $Q$  meets 3 facets of  $Q_3$ , but none of them multiply.



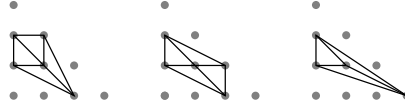
In the first case,  $\eta_T(1, 1) = 5$  and  $c_T(1, 1) = -7$ , and in the second case,  $\eta_T(1, 1) = 6$  and  $c_T(1, 1) = -6$ .

- Assume  $Q$  meets only two facets of  $Q_3$ , and none of them multiply.



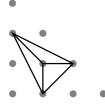
In the first case,  $\eta_T(1, 1) = 5$  and  $c_T(1, 1) = -7$ , and in the second case,  $\eta_T(1, 1) = 4$  and  $c_T(1, 1) = -8$ .

- Assume  $Q$  meets only one facet of  $Q_3$  and it does so non-multiply.



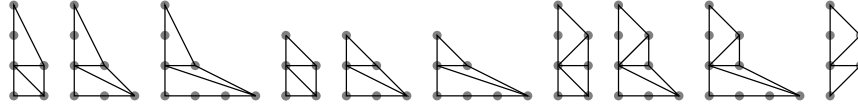
In the first and second case,  $\eta_T(1,1) = 4$  and  $c_T(1,1) = -8$ , and in the third case,  $\eta_T(1,1) = 3$  and  $c_T(1,1) = -9$ .

- Assume  $Q$  meets no facet of  $Q_3$  at all.

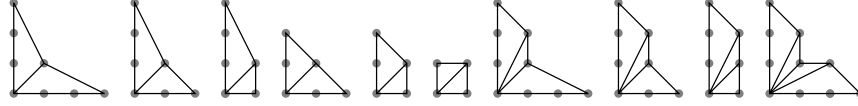


Finally, in this case,  $\eta_T(1,1) = 3$  and  $c_T(1,1) = -9$ .

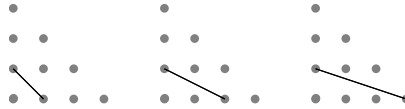
Thus the claim for  $(1,1)$  is shown. Now assume  $(1,1) \neq (i,j) \in Q$  is not a vertex of  $Q_3$ . If  $(i,j)$  is also not a vertex of any  $Q_\theta$  then there is no edge in the subdivision from  $(1,1)$  to  $(i,j)$  and thus  $(i,j)$  does not contribute to the formula for the cycle length, i.e.  $c_T(i,j) = 0$ . However, the same holds true also for  $\eta_T(i,j)$ . We may thus assume that  $(i,j)$  is a vertex of some  $Q_\theta$ , and we may without restriction assume  $(i,j) = (0,1)$ . The classification of cases we have to consider is very similar to the above, and we will not give the details – in particular, leaving the computation of  $c_T(i,j)$  and  $\eta_T(i,j)$  to the reader. We do not have to consider the whole of  $Q$ , but only the triangles which are adjacent to  $(i,j)$ .



If  $(1,1) \neq (i,j) \in Q$  is a vertex of  $Q_3$  (without restriction  $(i,j) = (0,0)$ ), the following cases have to be considered:



Finally, we have to consider the case where  $(i,j)$  is not part of  $Q$ . Obviously,  $c_T(i,j) = 0$  in this case and we have to show the same for  $\eta_T(i,j)$ . Assume first that  $(i,j)$  is a vertex of  $Q_3$ , without restriction we can assume  $(i,j) = (0,0)$ . There must be a facet of  $Q$  such that  $(0,0)$  is on one side of it and  $(1,1)$  is on the other. Then (up to symmetry) there are 3 possibilities for that facet.



Since we assumed that  $T$  is the representative with as few edges as possible, the triangle formed by that facet of  $Q$  and  $(0,0)$  can not be additionally subdivided in the second and third picture. In any case,  $(0,0)$  is a vertex of only one triangle, which has one facet of integer length 1 and one facet of integer length  $l$  where  $1 \leq l \leq 3$ . Thus  $\eta_T(0,0) = 1 - 1 - l + l = 0$ . Now assume that  $(i,j)$  is not a vertex of  $Q_3$ , without restriction  $(i,j) = (1,0)$ . Again there must be a facet of  $Q$  such

that  $(1, 0)$  is on one side and  $(1, 1)$  is on the other. Up to symmetry this can only be one of the line segments in the two right pictures above. We assumed that  $T$  is the representative of its  $\Delta$ -equivalence class with as few edges as possible. But that means there is no edge through  $(1, 0)$  and  $(1, 0)$  is not a vertex of a triangle in the subdivision. Thus  $\eta_T(1, 0) = 0$ .  $\square$

### Remark 5.6

In the proof above the computation that shows that  $\eta_T(i, j) = c_T(i, j)$  is different in each of the considered cases. In particular, in the computation of  $\eta_T(i, j)$  the part of  $Q_3$  which is not part of the cycle is involved while this is not the case for  $c_T(i, j)$ . Therefore it is most unfortunately not possible to replace the consideration of several cases by an argument which holds for all of them at the same time.

However, using `polymake` and `SINGULAR` one can compute the vertices of the Newton polytope of  $\Delta$  and for each vertex one can compute the dual cone in the Gröbner fan of  $\Delta$  and the triangulation of  $(Q_3, \mathcal{A}_3)$  with as few edges as possible corresponding to this cone. That way one can verify the above computations for  $c_T$  and  $\eta_T$ , since the values for  $\eta_T$  can be read off immediately from the exponents of the vertex of the Newton polytope, while the  $c_T$  can be computed with the formula in Lemma 3.17. These computations have been made using the procedure `displayFan` and the result can be obtained via the URL

<http://www.mathematik.uni-kl.de/~keilen/en/jinvariant.html>.

The advantage is that the file `discriminant_fan_of_cubic.ps` available via this url shows the cases not only up to symmetry, but it shows actually all possible cases.

## 6. DISCRIMINANT

The aim of this section is to show that the Gröbner fan of  $\Delta$  is a coarsening of the secondary fan. This follows from the Prime Factorization Theorem of Gelfand, Kapranov and Zelevinsky ([4, Chap. 10, Sect. 2], see also [1, Conj. 5.2]). We present our own proof because we hope that the techniques will be useful. They are similar to those of Sturmfels (see [16]).

Our reference for toric varieties and the  $\mathcal{A}$ -discriminant is [4], and we will use Notation 2.1. Let  $\mathcal{A} \subset \mathbb{Z}^A$  be a finite set of lattice points. It defines a projective toric variety  $X_{\mathcal{A}} \subset \mathbb{P}^{|\mathcal{A}|-1}$  over  $K$ , where  $K$  is any field with non-archimedean valuation  $\text{val} : K^* \rightarrow \mathbb{R}$  whose value group is dense in  $\mathbb{R}$  with respect to the Euclidean topology.

Our special case is  $\mathcal{A}_3 = Q_3 \cap \mathbb{Z}^2$ . In this case,  $X_{\mathcal{A}} = \mathbb{P}^2$  embedded in  $\mathbb{P}^9$  by the 3-uple embedding.

We review some facts about toric varieties. A Laurent polynomial with support  $\mathcal{A}$ ,

$$f = \sum_{\omega \in \mathcal{A}} a_{\omega} \underline{x}^{\omega}$$

can be thought of as defining a hypersurface in the toric variety  $X_{\mathcal{A}}$  (see [4]), and since this hypersurface coincides with the hyperplane section defined by the

coefficients of  $f$  under the embedding  $X_{\mathcal{A}} \subseteq \mathbb{P}^{|\mathcal{A}|-1}$  we identify a polynomial  $f = \sum_{\omega \in \mathcal{A}} a_{\omega} \cdot \underline{x}^{\omega}$  with the point  $[a_{\omega}]_{\omega \in \mathcal{A}}$  in the dual space  $(\mathbb{P}^{|\mathcal{A}|-1})^{\vee}$  of  $\mathbb{P}^{|\mathcal{A}|-1}$ .

The *dual variety*  $X_{\mathcal{A}}^{\vee} \subset (\mathbb{P}^{|\mathcal{A}|-1})^{\vee}$  is the Zariski closure of the set of all hyperplanes  $H$  such that  $X_{\mathcal{A}}^{\circ} \cap H$  is singular where  $X_{\mathcal{A}}^{\circ}$  is the open torus in  $X_{\mathcal{A}}$ . If  $X_{\mathcal{A}}^{\vee}$  is of codimension 1, then its defining polynomial is called the *discriminant*  $\Delta_{\mathcal{A}}$ , otherwise we say that the  $X_{\mathcal{A}}$  has a *degenerate dual variety* and we set  $\Delta_{\mathcal{A}} = 1$ . For our special case  $\mathcal{A}_3$ ,  $\Delta = \Delta_{\mathcal{A}_3}$  is the denominator of the  $j$ -invariant.

**Lemma 6.1**

*Suppose  $X_{\mathcal{A}}$  is smooth and  $f \in V(\Delta_{\mathcal{A}})$ , then the variety  $V(f)$ , considered as a hypersurface in  $X_{\mathcal{A}}$ , is singular.*

**Proof:**

Consider the universal hypersurface  $\mathcal{U} = \{\sum_{\omega \in \mathcal{A}} a_{\omega} \cdot \underline{x}^{\omega} = 0\} \subset (\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times X_{\mathcal{A}}$  and the flat subfamily defined by the vanishing of  $\Delta_{\mathcal{A}}$ :

$$\begin{array}{ccc} \{(f, x) \in \mathcal{U} \mid \Delta_{\mathcal{A}}(f) = 0\} & \hookrightarrow & V(\Delta_{\mathcal{A}}) \times X_{\mathcal{A}} \\ & \searrow & \downarrow \\ & & V(\Delta_{\mathcal{A}}) \end{array}$$

By definition the general fiber of this family is singular, and thus so must be each fiber.  $\square$

The fact, however, that this singular point need not be in the open torus is a problem when proving that the Gröbner fan of the discriminant  $\Delta_{\mathcal{A}}$  is a coarsening of the secondary fan of  $\mathcal{A}$ . For this we will have to reduce to the restriction of  $f$  to some face  $\Gamma$  which has a singular point in the torus orbit  $X_{\Gamma}^0$  (see Lemma 6.3).

Let  $Q_{\mathcal{A}} = \text{Conv}(\mathcal{A})$  and suppose  $\mathcal{A} = Q_{\mathcal{A}} \cap \mathbb{Z}^{\Lambda}$ . For each face  $\Gamma$  of  $Q_{\mathcal{A}}$  of dimension  $k$ , we have a parameterization of the open torus orbit  $X_{\Gamma}^0$  given by

$$i_{\Gamma} : (K^*)^k \hookrightarrow X_{\mathcal{A}}.$$

We may consider the restriction

$$f_{\Gamma} = \sum_{\omega \in \mathcal{A} \cap \Gamma} a_{\omega} \underline{x}^{\omega}$$

of  $f$  to  $X_{\Gamma}^0$  as a function

$$f_{\Gamma} : (K^*)^k \rightarrow K : \xi \mapsto f_{\Gamma}(i(\xi)).$$

Picking coordinates  $y_1, \dots, y_k$  on  $(K^*)^k$ , we define the subset  $Z_{\Gamma}$  of  $(\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times X_{\mathcal{A}}$  by

$$Z_{\Gamma} = \bigcup_{f \in (\mathbb{P}^{|\mathcal{A}|-1})^{\vee}} \{f\} \times i_{\Gamma} \left( V(f_{\Gamma}) \cap V\left(\frac{\partial f_{\Gamma}}{\partial y_1}\right) \cap \dots \cap V\left(\frac{\partial f_{\Gamma}}{\partial y_k}\right) \right).$$

$Z_{\Gamma}$  can be viewed as the subset of  $(\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times X_{\mathcal{A}}$  consisting of pairs  $(f, x)$  of functions  $f$  and points  $x$  such that  $f_{\Gamma}$  is singular on  $X_{\Gamma}^0$  at  $x$ . Note that if  $\Gamma$  is just a point, then  $X_{\Gamma}^0$  is a closed point of  $X_{\mathcal{A}}$ , and  $f$  is singular on  $X_{\Gamma}^0$  if and only if it is zero on  $X_{\Gamma}^0$ .

**Lemma 6.2**

For two faces  $\Gamma'$  and  $\Gamma$  of  $Q_{\mathcal{A}}$  with  $\Gamma' \subset \Gamma$  there is the following inclusion

$$\overline{Z_{\Gamma}} \cap \left( (\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times X_{\Gamma'}^0 \right) \subseteq Z_{\Gamma'}^0,$$

where  $\overline{Z_{\Gamma}}$  denotes the Zariski closure of  $Z_{\Gamma}$  in  $(\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times X_{\mathcal{A}}$ .

**Proof:**

We may restrict to the toric variety  $X_{\Gamma}$  and therefore, we may suppose that  $\Gamma = Q_{\mathcal{A}}$ . By applying a monomial change-of-variables (i.e. a change of variables of the form  $y_{\nu} = \prod_{\lambda \in \Lambda} x_{\lambda}^{a_{\nu\lambda}}$  for  $\nu \in \Lambda$  where  $(a_{\nu\lambda})_{\nu, \lambda \in \Lambda}$  is an integer matrix of determinant  $\pm 1$ ), we may suppose that  $\Gamma'$  lies in a coordinate subspace of  $\mathbb{R}^{\Lambda}$  given by  $x_{\lambda} = 0$  for  $\lambda \in \Lambda \setminus \Lambda'$  with  $\#\Lambda' = \dim(\Gamma')$ .

By construction, there is an embedding  $i : X_{\mathcal{A}} \hookrightarrow \mathbb{P}^{|\mathcal{A}|-1}$  of  $X_{\mathcal{A}}$  into projective space. Let us pick homogeneous coordinates  $[b_{\omega}]_{\omega \in \mathcal{A}}$  so that on the open torus  $X_{\mathcal{A}}^0 = (K^*)^{\Lambda}$  the map  $i$  is given by

$$(K^*)^{\Lambda} \longrightarrow \mathbb{P}^{|\mathcal{A}|-1} : \underline{x} \mapsto b_{\omega} = \underline{x}^{\omega}.$$

Now consider the following morphism

$$j = \text{id} \times i : (\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times X_{\mathcal{A}} \rightarrow (\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times \mathbb{P}^{|\mathcal{A}|-1},$$

which is the identity on the first factor and  $i$  on the second factor. We denote the coordinates on  $(\mathbb{P}^{|\mathcal{A}|-1})^{\vee}$  again by  $[a_{\omega}]_{\omega \in \mathcal{A}}$ . There is a universal hypersurface  $\mathcal{V} \subset (\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times \mathbb{P}^{|\mathcal{A}|-1}$  cut out by

$$\sum_{\omega \in \mathcal{A}} a_{\omega} \cdot b_{\omega} = 0.$$

Observe that  $\mathcal{U} = j^{-1}(\mathcal{V})$  is the universal hypersurface on  $(\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times X_{\mathcal{A}}$  defined by the vanishing of  $\sum_{\omega \in \mathcal{A}} a_{\omega} \cdot \underline{x}^{\omega}$ . We may also define a universal singular locus  $\mathcal{V}_{\text{Sing}} \subset (\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times \mathbb{P}^{|\mathcal{A}|-1}$  by the  $\#\Lambda + 1$  equations

$$\sum_{\omega \in \mathcal{A}} a_{\omega} \cdot b_{\omega} = \sum_{\omega \in \mathcal{A}} \omega_{\lambda} \cdot a_{\omega} \cdot b_{\omega} = 0 \quad \text{for } \lambda \in \Lambda.$$

It is straight forward from the definitions that

$$j^{-1}(\mathcal{V}_{\text{Sing}}) \cap ((\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times X_{\Gamma'}^0) = Z_{\Gamma'}.$$

Therefore,  $\overline{Z_{\Gamma}} \subseteq j^{-1}(\mathcal{V}_{\text{Sing}})$ .

We may define  $\mathcal{V}_{\Gamma', \text{Sing}} \subseteq (\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times \mathbb{P}^{|\mathcal{A}|-1}$  by

$$\sum_{\omega \in \Gamma' \cap \mathcal{A}} a_{\omega} \cdot b_{\omega} = \sum_{\omega \in \Gamma' \cap \mathcal{A}} \omega_{\lambda} \cdot a_{\omega} \cdot b_{\omega} = 0 \quad \text{for } \lambda \in \Lambda'.$$

Again we get immediately from the definition that

$$Z_{\Gamma'} = j^{-1}(\mathcal{V}_{\Gamma', \text{Sing}}) \cap ((\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times X_{\Gamma'}^0) = j^{-1}(\mathcal{X}) \cap ((\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times X_{\Gamma'}^0),$$

where

$$\mathcal{X} = \mathcal{V}_{\Gamma', \text{Sing}} \cap ((\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times V(b_{\omega} \mid \omega \notin \Gamma')) = \mathcal{V}_{\text{Sing}} \cap ((\mathbb{P}^{|\mathcal{A}|-1})^{\vee} \times V(b_{\omega} \mid \omega \notin \Gamma')).$$



Thus by taking inverse images by  $j$ , we get

$$\begin{aligned} \overline{Z_\Gamma} \cap ((\mathbb{P}^{|\mathcal{A}|-1})^\vee \times X_{\Gamma'}^0) &\subseteq j^{-1}(\mathcal{V}_{\text{Sing}}) \cap ((\mathbb{P}^{|\mathcal{A}|-1})^\vee \times X_{\Gamma'}^0) \\ &= j^{-1}(\mathcal{X}) \cap ((\mathbb{P}^{|\mathcal{A}|-1})^\vee \times X_{\Gamma'}^0) = Z_{\Gamma'}. \end{aligned}$$

□

**Lemma 6.3**

If  $f \in V(\Delta_{\mathcal{A}})$ , then there is some face  $\Gamma$  of  $Q_{\mathcal{A}}$  so that  $f_\Gamma$  is singular on  $X_\Gamma^0$ .

**Proof:**

If  $\mathcal{F}(Q_{\mathcal{A}})$  denotes the set of all faces of  $Q_{\mathcal{A}}$  including  $Q_{\mathcal{A}}$  itself, we have to show that

$$V(\Delta_{\mathcal{A}}) \subseteq \bigcup_{\Gamma \in \mathcal{F}(Q_{\mathcal{A}})} \pi(Z_\Gamma) = \pi \left( \bigcup_{\Gamma \in \mathcal{F}(Q_{\mathcal{A}})} Z_\Gamma \right), \quad (5)$$

where  $\pi : (\mathbb{P}^{|\mathcal{A}|-1})^\vee \times X_{\mathcal{A}} \rightarrow (\mathbb{P}^{|\mathcal{A}|-1})^\vee$  denotes the projection onto the first factor. Since by definition  $V(\Delta_{\mathcal{A}})$  is the Zariski closure of  $\pi(Z_{Q_{\mathcal{A}}})$  it suffices to show that the right hand side in (5) is Zariski closed, or equivalently that  $\bigcup_{\Gamma \in \mathcal{F}(Q_{\mathcal{A}})} Z_\Gamma$  is so. This, however, follows once we know that

$$\overline{Z_\Gamma} \subseteq \bigcup_{\Gamma' \in \mathcal{F}(Q_{\mathcal{A}})} Z_{\Gamma'} \quad (6)$$

for all  $\Gamma \in \mathcal{F}(Q_{\mathcal{A}})$ , where  $\overline{Z_\Gamma}$  denotes the Zariski closure of  $Z_\Gamma$ . Since the Zariski closure of  $X_\Gamma^0$  is  $\overline{X_\Gamma^0} = \bigcup_{\Gamma' \in \mathcal{F}(\Gamma)} X_{\Gamma'}^0$ , and  $\overline{Z_\Gamma} \subseteq \overline{X_\Gamma^0}$ , this fact follows from the Lemma 6.2. □

**Lemma 6.4**

Let  $g \in \mathbb{C}[y_1, \dots, y_k]$  be a polynomial with  $k+1$  terms whose Newton polytope  $N(g)$  is a  $k$ -dimensional simplex. Then the hypersurface  $V(g)$  has no singular points in  $(\mathbb{C}^*)^k$ .

**Proof:**

We may divide  $g$  by a monomial without changing  $V(g) \cap (\mathbb{C}^*)^k$ . Therefore, we may suppose that one vertex of  $N(g)$  is at the origin. By applying a monomial change-of-variables, i.e. a coordinate change of the form  $y_i \mapsto y_1^{a_{i1}} \cdots y_k^{a_{ik}}$  with  $a_{ij} \in \mathbb{Z}$  and  $\det((a_{ij})_{i,j=1,\dots,k}) = \pm 1$ , we may suppose that the edges from 0 to the other  $k$  vertices of  $N(g)$  lie along the axes. Therefore,

$$g = a + \sum_{i=1}^k a_i y_i^{c_i}$$

for  $c_i \in \mathbb{N}$ . In that case  $\frac{\partial f}{\partial y_i}$  is a monomial and has no root in  $(\mathbb{C}^*)^A$ . □

**Proposition 6.5**

Let  $\mathcal{A} \subset \mathbb{Z}^\Lambda$  be such that  $X_{\mathcal{A}}^\vee$  has codimension one and the discriminant  $\Delta_{\mathcal{A}}$  exists. Then the tropicalization  $\text{Trop}(V(\Delta_{\mathcal{A}}))$  of  $V(\Delta_{\mathcal{A}})$  is supported on the codimension one cones of the secondary fan of  $\mathcal{A}$  in  $\mathbb{R}^\Lambda$ .

**Proof:**

We assume that  $\text{Trop}(V(\Delta_{\mathcal{A}}))$  intersects the interior of a top dimensional cone of the secondary fan of  $\mathcal{A}$  in a point  $u \in \mathbb{Q}^{\mathcal{A}}$  and derive a contradiction.

Since  $u \in \text{Trop}(V(\Delta_{\mathcal{A}}))$  by the Lifting Lemma for hypersurfaces (see e.g. [2, Thm. 2.13]) we can lift  $u$  to a point  $f = [a_{\omega}]_{\omega \in \mathcal{A}} \in V(\Delta_{\mathcal{A}})$  such that  $\text{val}(a_{\omega}) = u_{\omega}$  for all  $\omega \in \mathcal{A}$  – in particular,

$$a_{\omega} \neq 0 \quad \text{for all } \omega \in \mathcal{A}. \quad (7)$$

By Lemma 6.3 there exists then a, say  $k$ -dimensional, face  $\Gamma$  of  $Q_{\mathcal{A}} = \text{Conv}(\mathcal{A})$  such that  $V(f_{\Gamma}) \subset (K^*)^k$  has a singular point  $\xi \in (K^*)^k$ , and  $k > 0$  due to (7). We define  $\omega = (\text{val}(\xi_1), \dots, \text{val}(\xi_k))$  to be the valuation of  $\xi$ , and we then claim that  $\xi_0 = (\text{lc}(\xi_1), \dots, \text{lc}(\xi_k)) \in (\mathbb{C}^*)^k$  is a singular point of  $\text{t-in}_{\omega}(f_{\Gamma})$ . In order to see that  $\xi_0$  is a singular point of  $\text{t-in}_{\omega}(f_{\Gamma})$  it suffices to note that

$$\text{t-in}_{\omega} \left( \frac{\partial f_{\Gamma}}{\partial x_i} \right) = \frac{\partial \text{t-in}_{\omega}(f_{\Gamma})}{\partial x_i},$$

and that for any polynomial  $g \in K[y_1, \dots, y_k]$  with  $g(\xi) = 0$  we necessarily have  $\text{t-in}_{\omega}(g)(\xi_0) = 0$ .

Since  $u$  is in the interior of a full-dimensional cone of the secondary fan of  $\mathcal{A}$  the Newton subdivision, say  $\{(Q_{\theta}, \mathcal{A}_{\theta}) \mid \theta \in \Theta\}$ , of  $f$  is a triangulation. By definition the Newton polytope of the  $t$ -initial form  $\text{t-in}_{\omega}(f_{\Gamma})$  is a face of some  $Q_{\theta}$  and is thus a simplex. But then by Lemma 6.4  $\text{t-in}_{\omega}(f_{\Gamma})$  has no singular point in the torus  $(\mathbb{C}^*)^k$  in contradiction to the existence of  $\xi_0$ .  $\square$

7. NUMERATOR OF THE  $j$ -INVARIANT

Unfortunately, for the numerator  $A$  of the  $j$ -invariant it is not true that the Gröbner fan of  $A$  is a coarsening of the secondary fan, as follows from Example 7.1.

**Example 7.1**

We provide an example which shows that the Gröbner fan of  $A$  is not a coarsening of the secondary fan in the case of curves of a particular form. The case of the full cubic is more complicated but analogous. It can easily be proved by a computation using `polymake` – this can be done using the procedure `nonrefinementC` in the library `jInvariant.lib` (see [8]).

Let us consider curves in Weierstrass form

$$y^2 + axy - x^3 - bx^2 - 1 = 0.$$

This corresponds to taking  $\mathcal{A} = \{(0, 2), (1, 1), (3, 0), (2, 0), (0, 0)\}$ . The fixing the constant coefficient and the coefficients of  $y^2$  and  $x^3$  has the effect of fixing an isomorphism  $\mathbb{R}^2 \cong R^{|\mathcal{A}|}/L$  in light of Remark 3.8.

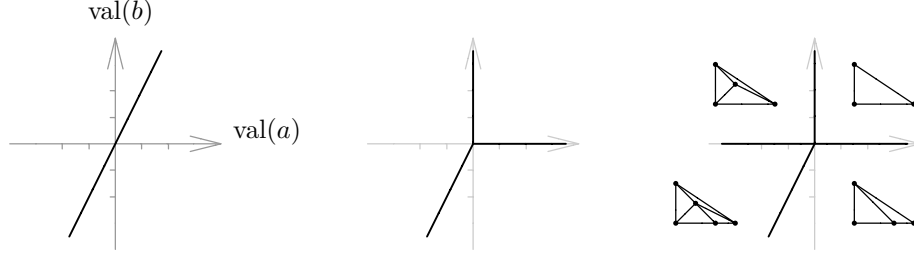
By the usual formulas for the  $j$ -invariant, we have

$$A = (a^2 + 4b)^6 \quad \text{and} \quad \Delta = -(a^2 + 4b)^3 - 432,$$

so that

$$j = -\frac{(a^2 + 4b)^6}{(a^2 + 4b)^3 + 432}.$$

The following picture shows the tropicalization of the numerator  $A$ , the tropicalization of the denominator  $\Delta$ , and the secondary fan in  $\mathbb{R}^{|\mathcal{A}|}/L$ .



Observe that the tropicalization of the denominator is supported on the codimension one skeleton of the secondary fan while that of the numerator intersects a top-dimensional cone of the secondary fan.

However, we are only interested in plane tropical cubics which have a cycle, that is, which are dual to marked subdivisions for which the interior point can be seen. All these cones of the secondary fan are completely contained in one cone of the Gröbner fan of  $A$ . We verified this computationally using `polymake` (see [3]). As usual we use the coordinates  $u_{ij}$  with  $(i, j) \in \mathcal{A}_3$  on  $\mathbb{R}^{\mathcal{A}_3}$  and we denote by  $e_{kl} = (\delta_{ik} \cdot \delta_{jl} \mid (i, j) \in \mathcal{A}_3)$  the canonical basis vector in  $\mathbb{R}^{\mathcal{A}_3}$  having a one in position  $kl$  and zeros elsewhere.

### Lemma 7.2

Let  $U$  be the union of all cones of the secondary fan of  $\mathcal{A}_3$  corresponding to marked subdivisions  $T = \{(Q_\theta, \mathcal{A}_\theta) \mid \theta \in \Theta\}$  of  $(Q_3, \mathcal{A}_3)$  for which  $(1, 1)$  is a vertex of some  $Q_\theta$ . Then  $U$  is contained in a single cone of the Gröbner fan of the  $A$ , namely in the cone dual to the vertex  $12e_{11}$  of the Newton polytope of  $A$ .

### Proof:

As input for `polymake` we use all exponents of the polynomial  $A \in \mathbb{Q}[\underline{a}]$ . The convex hull of the set of all exponents is the Newton polytope, say  $N(A)$ , of  $A$  and its vertices are the output of `polymake`. The Newton polytope has 19 vertices. Dual to each vertex is a top-dimensional cone of the Gröbner fan  $\mathcal{F}(A)$  of  $A$ , because the Gröbner fan is dual to the Newton polytope (see [17, Thm. 2.5 and Prop. 2.8]). The inequalities describing the cone  $C$  dual to the vertex  $V$  are given by the hyperplanes orthogonal to the edge vectors connecting  $V$  with its neighboring vertices in  $N(A)$ . We compute the neighboring vertices for each vertex using `polymake` and deduce thus inequalities for each of the top-dimensional cones of the Gröbner fan of  $A$ .

We do these computations identifying  $\mathbb{R}^{\mathcal{A}_3}$  with  $\mathbb{R}^{10}$  via the following ordering of the variables:

$$u_{11}, u_{30}, u_{20}, u_{10}, u_{00}, u_{21}, u_{01}, u_{12}, u_{02}, u_{03}.$$

In order for a marked subdivision  $T_\psi = \{(Q_\theta, \mathcal{A}_\theta) \mid \theta \in \Theta\}$  of  $(Q_3, \mathcal{A}_3)$  given by

$$\psi : \mathbb{R}^{\mathcal{A}_3} \rightarrow \mathbb{R} : (i, j) \mapsto u_{ij}$$

to have the point  $(1, 1)$  as vertex of some  $Q_\theta$  it is obviously necessary that the  $u_{ij}$  satisfy the following inequalities:

$$\begin{array}{ll}
3 \cdot u_{01} + 2 \cdot u_{30} + u_{03} > 6 \cdot u_{11} & 3 \cdot u_{10} + 2 \cdot u_{03} + u_{30} > 6 \cdot u_{11} \\
3 \cdot u_{12} + u_{30} + 2 \cdot u_{00} > 6 \cdot u_{11} & 3 \cdot u_{21} + u_{03} + 2 \cdot u_{00} > 6 \cdot u_{11} \\
2 \cdot u_{30} + 3 \cdot u_{02} + u_{00} > 6 \cdot u_{11} & 2 \cdot u_{03} + 3 \cdot u_{20} + u_{00} > 6 \cdot u_{11} \\
u_{12} + u_{30} + u_{00} + u_{02} > 4 \cdot u_{11} & u_{21} + u_{03} + u_{00} + u_{20} > 4 \cdot u_{11} \\
u_{01} + u_{10} + u_{03} + u_{30} > 4 \cdot u_{11} & 2 \cdot u_{01} + u_{12} + u_{30} > 4 \cdot u_{11} \\
2 \cdot u_{10} + u_{21} + u_{03} > 4 \cdot u_{11} & 2 \cdot u_{12} + u_{20} + u_{00} > 4 \cdot u_{11} \\
2 \cdot u_{21} + u_{02} + u_{00} > 4 \cdot u_{11} & 2 \cdot u_{02} + u_{10} + u_{30} > 4 \cdot u_{11} \\
2 \cdot u_{20} + u_{01} + u_{03} > 4 \cdot u_{11} & u_{20} + u_{01} + u_{12} > 3 \cdot u_{11} \\
u_{02} + u_{10} + u_{21} > 3 \cdot u_{11} & u_{30} + u_{02} + u_{01} > 3 \cdot u_{11} \\
u_{03} + u_{10} + u_{20} > 3 \cdot u_{11} & u_{00} + u_{12} + u_{21} > 3 \cdot u_{11} \\
u_{00} + u_{30} + u_{03} > 3 \cdot u_{11} & u_{21} + u_{01} > 2 \cdot u_{11} \\
u_{10} + u_{12} > 2 \cdot u_{11} & u_{20} + u_{02} > 2 \cdot u_{11}
\end{array} \tag{8}$$

These inequalities determine thus a cone in  $\mathbb{R}^{\mathcal{A}_3}$  which contains  $U$ . A simple computation with `polymake` allows to compute the extreme rays of this cone and to check that they actually satisfy the inequalities of the single cone of the Gröbner fan of  $A$  which is dual to the vertex  $12e_{11}$  in  $N(A)$ . This proves the claim.  $\square$

### Remark 7.3

The computations were done with the procedure `testInteriorInequalities` in the library `jinvariant.lib` (see [8]).

The inequalities in (8) actually determine precisely the cone  $U$ , which is less obvious than that they are necessary, but this can again be easily tested using `polymake`.

## 8. RAYS OF THE SECONDARY FAN

In this section, we classify the rays of the secondary fan of  $\mathcal{A}_3$ . This is part of our second proof of the main theorem.

### Definition 8.1

Let  $\nu \in \mathcal{A}_3$  be a lattice point that is not a vertex. The *lift* associated to  $\nu$  is the ray in  $\mathbb{R}^{\mathcal{A}_3}$  consisting of all functions  $\psi : \mathcal{A}_3 \rightarrow \mathbb{R}$  of the form

$$\psi(\omega) = \begin{cases} a + v \cdot \omega + b, & \text{if } \omega = \nu, \\ a + v \cdot \omega, & \text{else,} \end{cases}$$

for some  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}^+$ , and  $v \in \mathbb{R}^2$ . The marked subdivision of  $(Q_3, \mathcal{A}_3)$  associated to the lift is  $\{(Q_3, \mathcal{A}_3 \setminus \{\nu\})\}$ .

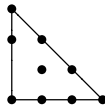


FIGURE 4. The subdivision associated to the lift with  $\nu = (0, 1)$ .

**Definition 8.2**

Let  $w \in \mathbb{R}^2$  and  $c \in \mathbb{R}$  be such that  $\{\omega \in \mathbb{R}^2 \mid \omega \cdot w = c\}$  is a line  $l$  through  $Q_3$  that intersects the boundary of  $Q_3$  in lattice points. The *fold through  $l$*  is the cone consisting of all functions  $\psi : \mathcal{A}_3 \rightarrow \mathbb{R}$  of the form

$$\psi(\omega) = a + v \cdot \omega + b \cdot \max\{\omega \cdot w - c, 0\}$$

for some  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}^+$ , and  $v \in \mathbb{R}^2$ . The subdivision associated to the fold is  $\{(Q_+, \mathcal{A}_+), (Q_-, \mathcal{A}_-)\}$  where

$$Q_+ = \{v \in Q_3 \mid v \cdot w \geq c\}$$

$$Q_- = \{v \in Q_3 \mid v \cdot w \leq c\}$$

and  $\mathcal{A}_+ = Q_+ \cap \mathcal{A}_3$ ,  $\mathcal{A}_- = Q_- \cap \mathcal{A}_3$ .

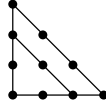


FIGURE 5. A subdivision associated to a fold.

**Definition 8.3**

Consider three lattice points  $p_1$ ,  $p_2$  and  $p_3$  on the boundary of  $Q_3$  (ordered counterclockwise) and the corresponding vectors  $v_i$  pointing from the interior point  $p = (1, 1)$  to  $p_i$ . The half lines starting in  $p$  and passing through  $p_i$  divide  $\mathbb{R}^2$  into three cones, say  $C_{12}$ ,  $C_{23}$  and  $C_{31}$  where  $C_{ij}$  contains the points  $p_i$  and  $p_j$  in its boundary. Note that the angle between  $v_i$  and  $v_{i+1}$  is less than 180 degrees. Define the function  $\phi : \mathcal{A}_3 \rightarrow \mathbb{R}$  by

$$\phi_b(\omega) = \begin{cases} 0 & \text{if } \omega \in C_{12} \\ tb & \text{if } \omega = p + sv_2 + tv_3 \in C_{23} \\ tb & \text{if } \omega = p + tv_3 + sv_1 \in C_{31} \end{cases}$$

for  $b \in \mathbb{R}^+$ . In other words  $\phi_b$  is 0 on  $C_1$  and is linear on  $p + v_3 \cdot \mathbb{R}^+$ . The *pinwheel* through  $p_1$ ,  $p_2$  and  $p_3$  is the cone consisting of all functions  $\psi : \mathcal{A}_3 \rightarrow \mathbb{R}$  of the form

$$\psi(\omega) = a + \omega \cdot v + \phi_b(\omega),$$

for some  $a \in \mathbb{R}$ ,  $b \in \mathbb{R}^+$  and  $v \in \mathbb{R}^2$ . The subdivision associated to the pinwheel is subdivision  $\{(Q_{12}, \mathcal{A}_{12}), (Q_{23}, \mathcal{A}_{23}), (Q_{31}, \mathcal{A}_{31})\}$  with  $Q_{ij} = C_{ij} \cap Q_3$  and  $\mathcal{A}_{ij} = Q_{ij} \cap \mathcal{A}_3$ .

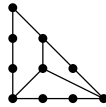


FIGURE 6. A subdivision associated to a pinwheel.

**Proposition 8.4**

*Any ray of the secondary fan of  $\mathcal{A}_3$  is a lift, a fold, or a pinwheel.*

**Proof:**

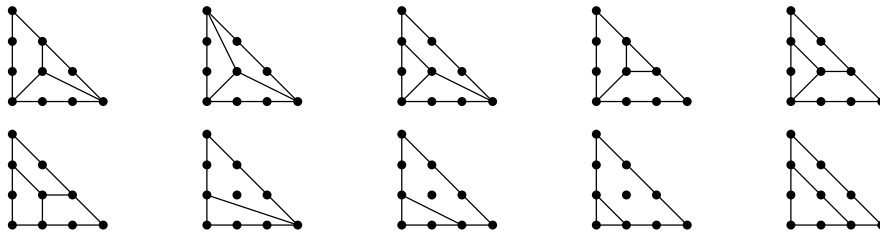
Note that lifts, folds and pinwheels are obviously rays of the secondary fan of  $\mathcal{A}_3$ .

Let  $C$  be a ray in the secondary fan of  $\mathcal{A}_3$ . Consider the associated subdivision  $S = \{(Q_\theta, \mathcal{A}_\theta) \mid \theta \in \Theta\}$ . The only non-trivial coarsening of  $S$  is the coarsest subdivision.

If  $\bigcup_{\theta \in \Theta} \mathcal{A}_\theta \neq \mathcal{A}_3$ , let  $\omega \in \mathcal{A}_3 \setminus \bigcup_{\theta \in \Theta} \mathcal{A}_\theta$ . The marked subdivision associated to the lift of  $\omega$  is a coarsening of  $S$  and the lift of  $\omega$  is a ray of the secondary fan of  $\mathcal{A}_3$ . Therefore the marked subdivision  $S$  itself must determine a lift.

Suppose  $\bigcup_{\theta \in \Theta} \mathcal{A}_\theta = \mathcal{A}_3$ . There must be at least 2 polygons in the subdivision. There must therefore be an edge  $E$  of a  $Q_\theta$  which is not an edge of  $Q_3$ . If the edge has both end-points on the boundary of  $Q_3$  then the fold through  $E$  is a coarsening of  $S$ . Therefore  $S$  must determine a fold. Otherwise, one end-point of  $E$  must be the interior lattice point,  $p$ . There can be at most 3 edges of the subdivision meeting at  $p$ . If there were more, we could remove all but 3 of them and still have less than 180 degrees counterclockwise between any two edges. Therefore, we have a pinwheel.  $\square$

There is a natural action of  $S_3 \subset \mathrm{PGL}_3(\mathbb{K})$  on  $\mathrm{Sym}^3(\mathbb{K}^3)$ . This induces an action of  $S_3$  on the secondary fan of  $\mathcal{A}_3$ . We list the folds and pinwheels that are rays of the secondary fan of  $\mathcal{A}_3$  up to  $S_3$  action.



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